Math 2233.21569

SOLUTIONS TO SECOND EXAM

April 1, 2021

1. (5 pts) Explain in words and formulas how you would construct the general solution of $y''+p(x)\,y'+q(x)\,y=g(x)$, given that $y_1(x)$ is a solution of $y''+p(x)\,y'+q(x)\,y=0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

- Step 1: Use Reduction of Order to find a second, independent, solution of the homogenous equation:
$$y_2 = y_1(x)\int \frac{1}{(y_1(x))^2}\exp\left(-\int^x p(s)\,ds\right)dx$$
- Step 2: Use Variations of Parameters to find a particular solution $y_p(x)$ of the inhomogeneous equation
$$y_p(x) = -y_1(x)\int \frac{y_2(x)\,g(x)}{W[y_1,y_2]}dx \quad + \quad y_2(x)\int \frac{y_1(x)\,g(x)}{W[y_1,y_2]}dx$$
- Step 3: The general solution of the inhomogeneous equation can now be constructed from y_1, y_2 and y_p :
$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x)$$
with c_1 and c_2 arbitrary constants.

2. (10 pts) Given that $y_1(x)=x^2$ is one solution of $x^2y''-3xy'+4y=0$, use Reduction of Order to determine the general solution of this differential equation.

- Putting the ODE in standard form, one has $y''-\frac{3}{x}y'+\frac{4}{x^2}y=0$ and so $p(x)=-\frac{3}{x}$. We now apply the Reduction of Order formula to calculate a second independent solution
$$\begin{aligned} y_2(x) &= y_1(x)\int \frac{1}{(y_1(x))^2}\exp\left[-\int p(x)\,dx\right]dx \\ &= x^2\int \frac{1}{x^4}\exp\left[-\int \frac{-3}{x}dx\right]dx \\ &= x^2\int \frac{1}{x^4}\exp\left[+3\ln|x|\right]dx = x^2\int \frac{1}{x^4}x^3dx \\ &= x^2\int \frac{1}{x}\,dx = x^2 \\ &= x^2\ln|x| \end{aligned}$$
The general solution is thus
$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x^2 + c_2x^2\ln|x|$$

3. Determine the general solution of the following differential equations.

(a) (5 pts) $y'' - 5y' + 6y = 0$

- This a constant coefficient ODE. Its characteristic equation is

$$0 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) \quad \Rightarrow \quad \lambda = 2, 3 \quad \Rightarrow \quad y_1(x) = e^{2x}\ ,\ y_2(x) = e^{3x}$$

and so

$$y(x0) = c_1 e^{2x} + c_2 e^{3x}$$

(b) (5 pt) $x^2 y'' + 5xy' - 12y = 0$

- This is an Euler type ODE. Its auxiliary equation is

$$0 = r^2 + (5-1)r - 12 = (\lambda + 6)(\lambda - 2) \quad \Rightarrow \quad \lambda = 2, -6 \quad \Rightarrow \quad y_1(x) = x^2\ ,\ y_2 = x^{-6}$$

and so

$$y(x) = c_1 x^2 + c_2 x^{-6}$$

(c) (5 pts) $x^2 y'' - xy' + y = 0$

- This is an Euler type ODE. Its auxiliary equation is

$$0 = r^2 + (-1-1)r + 1 = (r-1)^2 \quad \Rightarrow \quad r = 1 \quad \Rightarrow \quad y_1(x) = x^1\ ,\ y_2(x) = x^1 \ln|x|$$

and so

$$y(x) = c_1 x + c_2 x \ln|x|$$

(d) (5 pts) $y'' + 10y' + 25y = 0.$

- This is a constant coefficient ODE. Its characteristic equation is

$$0 = \lambda^2 + 10\lambda + 25 = (\lambda + 5)^2 \quad \Rightarrow \quad \lambda = -5 \quad \Rightarrow \quad y_1(x) = e^{-5x}\ ,\ y_2(x) = xe^{-5x}$$

and so

$$y(x) = c_1 e^{-5x} + c_2 x e^{-5x}$$

(e) (5 pts) $x^2 y'' + 5xy' + 5y = 0$

- This is an Euler type ODE. Its auxiliary equation is

$$0 = r^2 + (5-1)r + 5 = r^2 + 4r + 5 \quad \Rightarrow r = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i$$

Since we have two complex roots $\lambda = \alpha \pm i\beta$ with $\alpha = -2$, $\beta = 1$, the general solution will be

$$y(x) = c_1 x^{-2} \cos(x) + c_2 x^{-2} \sin(x)$$

(f) (5 pts) $y'' - 2y' + 10y = 0$

- This is a constant coefficients type ODE. Its characteristic equation is

$$0 = \lambda^2 - 2\lambda + 13 \quad \Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4 - 40}}{2} = 1 \pm 3i$$

Since we have two complex roots $\lambda = \alpha \pm i\beta$ with $\alpha = 1$, $\beta = 3$, the general solution will be

$$y\left(x\right) = c_1 e^x \cos\left(3x\right) + c_2 e^x \sin\left(3x\right)$$

4. Given that $y_1(x) = e^{-x}$ and $y_2(x) = e^{2x}$ are solutions of $y'' - y' - 2y = 0$:

(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of $y'' - y' - 2y = 6e^x$.

- The differential equation is in standard form with $g(x) = 6e^x$. We have

  $$W[y_1, y_2] = y_1 y_2' = y_1' y_2 = \left(e^{-x}\right)\left(2e^{2x}\right) - \left(-e^{-x}\right)\left(e^{2x}\right) = 3e^x$$

  Applying the Variation of Parameters formula

  $$\begin{aligned} y_p(x) &= -y_1 \int \frac{y_2 g}{W[y_1, y_2]} dx + y_2 \int \frac{y_1 g}{W[y_1, y_2]} dx \\ &= -e^{-x} \int \frac{e^{2x}(6e^x)}{3e^x} dx + e^{2x} \int \frac{e^{-x}(6e^x)}{3e^x} dx \\ &= -e^{-x} \int 2e^{2x} dx + e^{2x} \int 2e^{-x} dx \\ &= -e^{-x}\left(e^{2x}\right) + e^{2x}\left(-2e^{-x}\right) \\ &= -3e^x \end{aligned}$$

(b) (10 pts) Find the solution of the differential equation in part (a) satisfying $y(0) = -1$, $y'(0) = -8$.

- The general solution to the nonhomogeneous ODE will be

  $$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = -3e^x + c_1 e^{-x} + c_2 e^{2x}$$

  Imposing the initial conditions on the general solution we find

  $$\begin{aligned} -1 &= y(0) = -3 + c_1 + c_2 \\ -8 &= y'(0) = \left(-3e^x - c_1 e^{-x} + 2c_2 e^{2x}\right)\Big|_{x=0} = -3 - c_1 + 2c_2 \end{aligned}$$

  Solving these two linear equations for c_1 and c_2 yields

  $$c_1 = 3 \quad , \quad c_2 = -1$$

  and so our solution is

  $$y(x) = -3e^x + 3e^{-x} - e^{2x}$$

5. Invert the following Laplace Transforms (i.e find the function $f(t)$ with the given Laplace transform).

(a) (10 pts) $\mathcal{L}[f](s) = \dfrac{s+2}{s^2-2s-3}$

- $$\mathcal{L}[f](s) = \frac{s+2}{s^2-2s-3} = \frac{s+2}{(s-3)(s+1)}$$

  Setting up a Partial Fractions Expansion of the function on the right

  $$\frac{s+2}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \quad \Rightarrow \quad s+2 = (s+1)A + (s-3)B$$

  Using special values of s

  $$s = 3 \quad \Rightarrow \quad 5 = (4)A + (0)B \quad \Rightarrow \quad A = \frac{5}{4}$$
  $$s = -1 \quad \Rightarrow \quad 1 = (0)A + (-4)B \quad \Rightarrow \quad B = -\frac{1}{4}$$

  and so

  $$\mathcal{L}[f](s) = \frac{5}{4}\frac{1}{s-3} - \frac{1}{4}\frac{1}{s+1} = \frac{5}{4}\mathcal{L}\left[e^{3x}\right] - \frac{1}{4}\mathcal{L}\left[e^{-x}\right] = \mathcal{L}\left[\frac{5}{4}e^{3x} - \frac{1}{4}e^{-x}\right]$$

  and so

  $$f(x) = \frac{5}{4}e^{3x} - \frac{1}{4}e^{-x}$$

(b) (10 pts) $\mathcal{L}[f](s) = \dfrac{s}{s^2-2s+10}$ (Hint: try completing the square in the dominator)

- We have

  $$s^2 - 2s + 10 = s^2 - 2s + 1 - 1 + 10 = (s-1)^2 + 9 = (s-1)^2 + 3^2$$

  The denominator is thus a sum of squares of the form $(s-a)^2 + b^2$ with $a = 1$, $b = 3$. We'll thus try to express $\mathcal{L}[f](s)$ as a combination of the Laplace transforms of $\mathcal{L}[e^x \cos(3x)] = \frac{s-1}{(s-1)^2+3^2}$ and $\mathcal{L}[e^x \sin(3x)] = \frac{3}{(s-1)^2+3^2}$:

  $$\mathcal{L}[f](s) = \frac{s}{s^2-2x-10} = A\frac{s-1}{(s-1)^2+3^2} + B\frac{3}{(s-1)^2+3^2}$$
  $$\Rightarrow \quad s = (s-1)A + 3B$$

  But now

  $$s = 1 \quad \Rightarrow \quad 1 = 3B \quad \Rightarrow \quad B = \frac{1}{3}$$
  $$s = 0 \quad \Rightarrow \quad 0 = -A + 3B = -A + 1 \quad \Rightarrow \quad A = 1$$

  Thus,

  $$\mathcal{L}[f](s) = \frac{s-1}{(s-1)^2+3^2} + \frac{1}{3}\frac{3}{(s-1)^2+3^2}$$
  $$= \mathcal{L}[e^x \cos(3x)] + \frac{1}{3}\mathcal{L}[e^x \sin(3x)] = \mathcal{L}\left[e^x \cos(3x) + \frac{1}{3}e^x \sin(3x)\right]$$

and so

$$f(x) = e^x \cos(3x) + \frac{1}{3} e^x \sin(3x)$$

6. (15 pts) Solve the following initial value problems **using the Laplace transform method**.

$$y'' + 5y' + 6y = 0 \qquad ; \qquad y(0) = 2 \quad , \quad y'(0) = -6$$

- Taking the Laplace transform of the differential equation yields

$$\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 5\left(s\mathcal{L}[y] - y(0)\right) + 6\mathcal{L}[y] = 0$$

  or

$$\left(s^2 + 5s + 6\right)\mathcal{L}[y] - s(2) + 6 - (5)(2) = 0$$

  or

$$\mathcal{L}[y] = \frac{2s+4}{s^2+5s+6} = \frac{2(s+2)}{(s+2)(s+3)} = \frac{2}{s+3} = 2\mathcal{L}\left[e^{-3x}\right] = \mathcal{L}\left[2e^{-3x}\right]$$

  and so the solution to the initial value problem is

$$y(x) = 2e^{-3x}$$