Math 2233.21570
SOLUTIONS TO SECOND EXAM
April 1, 2021

1. (10 pts) Explain in words and formulas how you would construct the general solution of $y''+p(x)\,y'+q(x)\,y=g(x)$, given that $y_1(x)$ is a solution of $y''+p(x)\,y'+q(x)\,y=0$. (That is, describe the general procedure, writing down the relevant formulas. It is **not** necessary to carry out any calculations.)

- Step 1: Use Reduction of Order to find a second, independent, solution of the homogenous equation:
$$y_2 = y_1(x)\int \frac{1}{(y_1(x))^2}\exp\left(-\int^x p(s)\,ds\right)dx$$
- Step 2: Use Variations of Parameters to find a particular solution $y_p(x)$ of the inhomogeneous equation
$$y_p(x) = -y_1(x)\int \frac{y_2(x)\,g(x)}{W[y_1,y_2]}dx \quad + \quad y_2(x)\int \frac{y_1(x)\,g(x)}{W[y_1,y_2]}dx$$
- Step 3: The general solution of the inhomogeneous equation can now be constructed from y_1, y_2 and y_p :
$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x)$$
with c_1 and c_2 arbitrary constants.

2. (10 pts) Given that $y_1(x) = x^{-2}$ is one solution of $x^2y'' + 5xy' + 4y = 0$, use Reduction of Order to determine the general solution of this differential equation.

- Putting the ODE in standard form we find
$$y'' + \frac{5}{x}y' + \frac{4}{x^2}y = 0 \quad \Rightarrow \quad p(x) = \frac{5}{x}$$
We'll now apply the Reduction of Order formula
$$\begin{aligned} y_2(x) &= y_1(x)\int \frac{1}{(y_1)^2}\exp\left[-\int p(x)\,dx\right]dx \\ &= x^{-2}\int \frac{1}{x^{-4}}\exp\left[-\int \frac{5}{x}dx\right] \\ &= x^{-2}\int x^4\exp\left[-5\ln[x]\right] = x^{-2}\int x^4x^{-5}dx = x^{-2}\int x^{-1}dx \\ &= x^{-2}\ln|x| \end{aligned}$$
So a second independent solution is $y_2(x) = x^{-2}\ln|x|$ and the general solution is
$$y(x) = c_1x^{-2} + c_2x^{-2}\ln|x|$$

3. Determine the general solution of the following differential equations.

(a) (5 pts) $x^2y'' + 5xy' - 12y = 0$

- This is an Euler type ODE. Its auxilary equation is

  $$0 = r^2 + (5-1)\,r - 12 = r^2 + 4r - 12 = (r+6)\,(r-2) \quad \Rightarrow \quad r = -6, 2$$

  Since we have two real roots, the general solution will be

  $$y\,(x) = c_1x^{-6} + c_2x^2$$

(b) (5 pts) $y'' - 8y' + 16y = 0$.

- This is a constant coefficients ODE. Its characteristic equation is

  $$\lambda^2 - 8\lambda + 16 = (\lambda - 4)^2 \quad \Rightarrow \quad \lambda = 4$$

  Since we have only 1 real root, the general solution will be

  $$y\,(x) = c_1e^{4x} + c_2xe^{4x}$$

(c) (5 pts) $y'' - 2y' + 5y = 0$

- This is a constant coefficients ODE. Its characteristic equation is

  $$0 = \lambda^2 - 2\lambda + 5 \quad \Rightarrow \quad \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$

  Since we have a pair of complex roots $\lambda = \alpha \pm i\beta$ with $\alpha = 1$ and $\beta = 2$, the general soluion will be

  $$y\,(x) = c_1e^x \cos(2x) + c_2e^x \sin(2x)$$

(d) (5 pts) $x^2y'' - 3xy' + 5y = 0$

- This is an Euler type ODE. Its auxilary equation is

  $$0 = r^2 + (-3-1)\,r + 5 = r^2 - 4r + 5 \quad \Rightarrow \quad r = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

  Since we have a pair of complex roots $\lambda = \alpha \pm i\beta$ with $\alpha = 2$, $\beta = 1$, the general solution will be

  $$y\,(x) = c_1x^2 \cos(\ln|x|) + c_2x^2 \sin(\ln|x|)$$

(e) (5 pts) $x^2y'' - 3xy' + 4y = 0$

- This is an Euler type ODE. Its auxilary equation is

  $$r^2 + (-3-1)\,r + 4 = r^2 - 4r + 4 = (r-2)^2 \quad \Rightarrow \quad r = 2$$

  Since we have only one real root $r = 2$, the general solution will be

  $$y\,(x) = c_1x^2 + c_2x^2 \ln|x|$$

(f) (5 pts) $y'' + 9y = 0$

- This is a constant coefficients ODE. Its characteristic equation is

$$0 = \lambda^2 + 9 \quad \Rightarrow \quad \lambda = \pm 3i$$

Since we have a pair of complex roots $\lambda = \alpha \pm i\beta$ with $\alpha = 0$ and $\beta = 3$, the general solution will be

$$y(x) = c_1 e^{0x} \cos(3x) + c_2 e^{0x} \sin(3x) = c_1 \cos(3x) + c_2 \sin(3x)$$

4. Given that $y_1(x) = e^x$ and $y_2(x) = e^{-2x}$ are solutions of $y'' + y' - 2y = 0$:

(a) (10 pts) Use the Method of Variation of Parameters to find a particular solution of $y'' + y' - 2y = 12e^{2x}$.

- The differential equation is in standard form with $g(x) = 12e^{2x}$. Also
$$W[y_1, y_2] = y_1 y_2' - y_1' y_2 = (e^x)\left(-2e^{-2x}\right) - (e^x)\left(e^{-2x}\right) = -3e^{-x}$$
: Applying the Variation of Parameters formula
$$\begin{aligned} y_p(x) &= -y_1(x)\int \frac{y_2(x)\,g(x)}{W[y_1,y_2]}dx + y_2(x)\int \frac{y_1(x)\,g(x)}{W[y_1,y_2]}dx \\ &= -e^x \int \frac{e^{-2x}\left(12e^{2x}\right)}{-3e^{-x}}dx + e^{-2x}\int \frac{e^x\left(12e^{2x}\right)}{-3e^{-x}}dx \\ &= e^x \int 4e^x dx - e^{-2x}\int 4e^{4x}dx \\ &= 4e^x e^x - e^{-2x}e^{4x} \\ &= 3e^{2x} \end{aligned}$$

(b) (10 pts) Find the solution of the differential equation in part (a) satisfying $y(0) = 6$, $y'(0) = 3$.

- The general solution to the ODE is
$$y(x) = y_p(x) + c_1 y_1(x) + c_2 y_2(x) = 3e^{2x} + c_1 e^x + c_2 e^{-2x}$$
Imposing the initial conditions on the general solution yields
$$\begin{aligned} 6 &= y(0) = 3 + c_1 + c_2 \\ 3 &= y'(0) = \left(6e^{2x} + c_1 e^x - 2c_2 e^{-2x}\right)\Big|_{x=0} = 6 + c_1 - 2c_2 \end{aligned}$$
Solving these two equations for c_1 and c_2 yields $c_1 = 1$ and $c_2 = 2$. Thus, the solution ot the initial value problem is
$$y(x) = 3e^{2x} + e^x + 2e^{-2x}$$

5. Invert the following Laplace Transforms (i.e find the function $f(t)$ with the given Laplace transform).

(a) (10 pts) $\mathcal{L}[f](s) = \dfrac{s-3}{s^2-3s+2}$

- We have
$$\mathcal{L}[y] = \frac{s-3}{s^2-3s+2} = \frac{s-3}{(s-1)(s-2)}$$
Since the denominator factorizes, we'll look for a Partial Fractions Expansion of the right hand side
$$\frac{s-3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \quad \Rightarrow \quad s-3 = (s-2)A + (s-1)B$$
This last equation has to be true for all s. In particular,
$$\begin{aligned} s = 1 \quad &\Rightarrow \quad -2 = (-1)A + (0)B \quad \Rightarrow \quad A = 2 \\ s = 2 \quad &\Rightarrow \quad -1 = (0)A + (1)B \quad \Rightarrow \quad B = -1 \end{aligned}$$
Thus,
$$\mathcal{L}[f] = 2\frac{1}{s-1} - \frac{1}{s-2} = 2\mathcal{L}[e^x] - \mathcal{L}\left[e^{2x}\right] = \mathcal{L}\left[2e^x - e^{2x}\right]$$
And so
$$f(x) = 2e^x - e^{2x}$$

(b) (10 pts) $\mathcal{L}[f](s) = \dfrac{1}{s^2+2s+10}$ (Hint: try completing the square in the dominator)

- We have
$$\mathcal{L}[f] = \frac{1}{s^2+2s+1-1+10} = \frac{1}{(s+1)^2+3^2}$$
Note that the denominator is of the form $(s-a)^2+b^2$ with $a=-1$ and $b=3$ and so we'll try to view $\mathcal{L}[y]$ as being formed from Laplace transforms $\mathcal{L}[e^{-x}\cos(3x)] = \frac{s+1}{(s+1)^2+3^2}$ and $\mathcal{L}[e^{-x}\sin(3x)] = \frac{3}{(s+1)^2+3^2}$. Setting
$$\frac{1}{(s+1)^2+3^2} = A\frac{s+1}{(s+1)^2+3^2} + B\frac{3}{(s+1)^2+3^2} \quad \Rightarrow \quad 1 = (s+1)A + 3B$$
we see
$$\begin{aligned} s = -1 \quad &\Rightarrow \quad 1 = (0)A + 3B \quad \Rightarrow \quad B = \frac{1}{3} \\ s = 0 \quad &\Rightarrow \quad 1 = A + 3B = A + 1 \quad \Rightarrow \quad A = 0 \end{aligned}$$
and so
$$\mathcal{L}[f] = 0 + \frac{1}{3}\frac{3}{(s+1)^2+3^2} = \frac{1}{3}\mathcal{L}\left[e^{-x}\sin(3x)\right] = \mathcal{L}\left[\frac{1}{3}e^{-x}\sin(3x)\right]$$
Thus,
$$f(x) = \frac{1}{3}e^{-x}\sin(3x)$$

6. (20 pts) Solve the following initial value problems **using the Laplace transform method**.

$$y'' + 3y' - 4y = 0 \qquad ; \qquad y(0) = 2 \quad , \quad y'(0) = 2$$

- Taking the Laplace transform of the ODE yields

  $$\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 3\left(s\mathcal{L}[y] - y(0)\right) - 4\mathcal{L}[y] = 0$$

  or

  $$\left(s^2 + 3s - 4\right)\mathcal{L}[y] - 2s - 2 + (3)(-2) = 0$$

  or

  $$\mathcal{L}[y] = \frac{2s+8}{(s+4)(s-1)} = \frac{2(s+4)}{(s+4)(s-1)} = \frac{2}{s-1} = 2\mathcal{L}[e^x] = \mathcal{L}[2e^x]$$

  and so

  $$\mathcal{L}[y] = 2e^x$$