Math 2233.21570

SOLUTIONS TO FIRST EXAM

February 23, 2021

1. Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) $\dfrac{dx^2}{dt^2} + tx = \sin(x)$

- 2nd order, nonlinear, ODE

(b) $\dfrac{\partial^3 \psi}{\partial^3 x} - x^2 \dfrac{\partial \psi}{\partial y} = \psi^2$

- 3rd order, nonlinear, PDE

(c) $\dfrac{d^3 x}{dt^3} + t^2 \dfrac{dx}{dt} + x = 0$

- 3rd order, linear, ODE

(d) $x^2 y' + y = e^x \sin(x)$

- 1st order, linear, ODE

(e) $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial \phi}{\partial y} = (x+y)^2$

- 2nd order, linear, PDE

2. Consider the following first order ODE: $y' = xy$ and suppose $y(x)$ is the solution satisfying $y(1) = 2$. Use the numerical (Euler) method with $n = 3$ and $\Delta x = 0.1$ to estimate $y(1.3)$.

- From the problem statement we see that we have a differential equation of the form $\frac{dy}{dx} = F(x, y)$ with

$$F(x, y) = xy$$

We'll construct a table of approximate points (x_i, y_i) on the graph of the solution. The initial condition $y(1) = 2$, tell us that we should take

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 2 \end{aligned}$$

We'll now use Euler's Method to calculate subsequent pairs (x_i, y_i) until we reach $i = 3$.

$$\begin{aligned} x_1 &= x_0 + \Delta x = 1 + 0.1 = 1.1 \\ y_1 &= y_0 + F(x_0, y_0)\Delta x = y_0 + x_0 y_0 \Delta x = 2 + (1)(2)(0.1) = 2.2 \\ x_2 &= x_1 + \Delta x = 1.2 \\ y_2 &= y_1 + x_1 y_1 \Delta x = 2.2 + (1.1)(2.2)(0.1) = 2.442 \\ x_3 &= x_2 + \Delta x = 1.3 \\ y_3 &= y_2 + x_2 y_2 \Delta x = 2.442 + (1.2)(2.442)(0.1) = 2.735 \end{aligned}$$

Since $y_3 \approx y(x_3) = y(1.3)$, we conclude

$$y(1.3) \approx 2.735$$

3. Find an explicit solution of the following (separable) differential equation.

$$3x^2 - e^{2y}y' = x$$

- First, we must get the ODE in the explicitly separable form $M(x) + N(y)\frac{dy}{dx} = 0$. So we move the "$x$" term to the left hand side, and see that

$$M(x) = 3x^2 - x \quad , \quad N(y) = -e^{2y}$$

Next, we integrate $M(x)$ with respect to x and integrate $N(y)$ with respect to y, set the sum of these two integrations equal to a constant C :

$$\int (3x^2 - x)\, dx + \int (-e^{2y})\, dy = C$$

or

$$x^3 - \frac{1}{2}x^2 - \frac{1}{2}e^{2y} = C$$

or

$$\frac{1}{2}e^{2y} = x^3 - \frac{1}{2}x^2 - C$$

or

$$e^{2y} = 2x^3 - x^2 - 2C$$

or

$$y(x) = \frac{1}{2}\ln\left|2x^3 - x^2 - 2C\right|$$

4. Solve the following initial value problem

$$xy' - 2y = x^2 \qquad , \qquad y(1) = 2$$

- This is a first order linear ODE. To put it in the standard form $y' + p(x)y = g(x)$, we divide by x

$$y' - \frac{2}{x}y = x \quad \Rightarrow \quad p(x) = -\frac{2}{x} \quad , \quad g(x) = x$$

Next, we compute the integrating factor $\mu(x)$:

$$\mu(x) = \exp\left[\int p(x)\, dx\right] = \exp\left[\int -\frac{2}{x}dx\right] = \exp\left[-2\ln|x|\right] = x^{-2}$$

Now we can calculate the general solution:

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int \mu(x)\, g(x)\, dx + \frac{C}{\mu(x)} \\ &= x^2 \int (x^{-2})(x)\, dx + Cx^2 \\ &= x^2 \int \frac{1}{x}dx + Cx^2 \\ &= x^2 \ln|x| + Cx^2 \end{aligned}$$

To find the solution satisfying the initial condition $y(1) = 2$, we impose this condition on the general solution:

$$\begin{aligned} 2 &= y(1) = (1^2)\ln|1| + C(1)^2 = 0 + C \\ &\Rightarrow \quad C = 2 \end{aligned}$$

The solution to the initial value problem is thus

$$y(x) = x^2 \ln|x| + 2x^2$$

5. Consider the following initial value problem

$$\begin{aligned} 3x^2 + (2y - 2x)\frac{dy}{dx} &= 2y \\ y(1) &= 2 \end{aligned}$$

Show that the differential equation is exact and then find the explicit solution satisfying the initial condition.

- Putting the ODE in the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, we see

$$\begin{aligned} M(x,y) &= 3x^2 - 2y \\ N(x,y) &= 2y - 2x \end{aligned}$$

We have

$$\frac{\partial M}{\partial y} = -2 = \frac{\partial N}{\partial x}$$

and so the differential equation **is exact**. This means that its solutions coincide with the solutions of an algebraic equation

$$\Phi(x,y) = C$$

with $\Phi(x,y)$ determined by $\frac{\partial \Phi}{\partial x} = M(x,y)$, $\frac{\partial \Phi}{\partial y} = N(x,y)$. These conditions imply

$$\begin{aligned} \Phi(x,y) &= \int M(x,y)\,\partial x + c_1(y) = \int \left(3x^2 - 2y\right)\partial x + c_1(y) = x^3 - 2yx + c_1(y) \\ \Phi(x,y) &= \int N(x,y)\,\partial y + c_2(x) = \int (2y - 2x)\,\partial y + c_2(x) = y^2 - 2xy + c_2(x) \end{aligned}$$

To get these two expressions for $\Phi(x,y)$ to agree, we need to choose

$$c_1(y) = y^2 \qquad \text{and} \qquad c_2(x) = x^3$$

Thus,

$$\Phi(x,y) = x^3 - 2xy + y^2$$

Before solving $\Phi(x,y) = C$, we'll first find the value of C, for the given initial condition.

$$\begin{aligned} x &= 1 \text{ and } y = 2 \quad \Rightarrow \quad (1)^3 - 2(1)(2) + (2)^2 = C \\ &\Rightarrow \quad C = 1 \end{aligned}$$

Thus,

$$x^3 - 2xy + y^2 = C$$

This equation is quadratic in y, so we can solve it using the Quadratic Formula:

$$\begin{aligned} y &= \frac{2x \pm \sqrt{(-2x)^2 - 4(1)(x^3 - C)}}{2} \\ &= x \pm \sqrt{x^2 - x^3 + C} \end{aligned}$$

6. Find the general solution of

$$x\frac{dy}{dx} = x + y$$

(Hint: recast the differential equation into the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ and then use the change of variable $z(x) = \frac{y(x)}{x}$.)

- (Corrected Problem) Dividing both sides of the differential equation by x, we get

(*) $$\frac{dy}{dx} = 1 + \frac{y}{x}$$

Equation (*) is of the desired form with $F(z) = z + 1$. Let us now make a change of variable $z = \frac{y}{x}$. Then

$$y = zx \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z$$

If we now replace the $\frac{dy}{dx}$ on the left side of (*) by $x\frac{dz}{dx} + z$, and $\frac{x}{y}$ on the right side of (*) by z, we get

$$x\frac{dz}{dx} + z = z + 1$$

Cancelling the z term appearing on both sides, we get

$$x\frac{dz}{dx} = 1$$

or

$$-\frac{1}{x} + \frac{dz}{dx} = 0$$

which is a **separable** equation. The separable equation for $z(x)$ is readily solved:

$$-\int \frac{1}{x}dx + \int (1)\, dz = C$$

or

$$-\ln|x| + z = C \qquad \Rightarrow \qquad z = \ln|x| + C$$

Finally, we use $y = zx$, to get

$$y = xz = x\ln|x| + Cx$$

- (Original, Uncorrected Problem)

$$y\frac{dy}{dx} = x + y$$

Divide by x

$$\frac{dy}{dx} = \frac{x}{y} + 1 = F\left(\frac{y}{x}\right)$$

with

$$F(z) = \frac{1}{z} + 1 \quad .$$

Try the change of variables $z = \frac{y}{x} \quad \Rightarrow \quad y = zx \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z$. Making this change of variables

$$x\frac{dz}{dy} + z = \frac{dy}{dx} = \frac{x}{y} + 1 = \frac{1}{z} + 1$$

or

$$x\frac{dz}{dx} = \frac{1}{z} + 1 - z = \frac{1 + z - z^2}{z}$$

Dividing both sides by $\frac{1+z-z^2}{z}x$ yields

$$\frac{z}{1 + z - z^2}\frac{dz}{dx} = \frac{1}{x}$$

or

$$\frac{1}{x} + \frac{z}{z^2 - z - 1}\frac{dz}{dx}$$

which is a Separable Equation. Full credit was given for making it this far on the original problem.