Math 2233.21569
SOLUTIONS TO FIRST EXAM
February 23, 2021

1. Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) $\dfrac{\partial^2 \phi}{\partial t^2} - x^2 \dfrac{\partial^2 \phi}{\partial x^2} = 0$

- 2nd Order, Linear, PDE

(b) $\dfrac{dy}{dx} + x^2 y = e^x$

- 1st Order, Linear, ODE

(c) $\dfrac{d^2 x}{dt^2} + x \dfrac{dx}{dt} + x = 0$

- 2nd Order, Nonlnear, ODE

(d) $\dfrac{d^3 \phi}{dt^3} + t \dfrac{d\phi}{dt} = \sin(t)$

- 3rd Order, Linear, ODE

(e) $\dfrac{\partial^2 \phi}{\partial x^2} + y \left(\dfrac{\partial \phi}{\partial y} \right)^2 = 0$

- 2nd Order, Nonlinear, PDE

2. Consider the following first order ODE: $y' = xy + y$ and suppose $y(x)$ is the solution satisfying $y(0) = 1$. Use the numerical (Euler) method with $n = 3$ and $\Delta x = 0.1$ to estimate $y(0.3)$.

- From the problem statement we have

$$\begin{aligned} y' &= xy + y \quad \Rightarrow \quad F(x, y) = xy + y \\ y(0) &= 1 \quad \Rightarrow \quad x_0 = 0 \quad , \quad y_0 = 1 \\ \Delta x &= 0.1 \end{aligned}$$

and so

$$\begin{aligned} x_1 &= x_0 + \Delta x = 0 + 0.1 = 0.1 \\ y_1 &= y_0 + F(x_0, y_0)\,\Delta x = 1 + (x_0 y_0 + y_0)(0.1) = 1 + (0 + 1)(0.1) = 1.1 \\ x_2 &= x_1 + \Delta x = 0.2 \\ y_2 &= y_1 + (x_1 y_1 + y_1)\,\Delta x = 1.1 + ((0.1)(1.1) + 1.1)(0.1) = 1.221 \\ x_3 &= x_2 + \Delta x = 0.3 \\ y_3 &= y_2 + (x_2 y_2 + y_2)\,\Delta x = 1.221 + ((0.2)(1.221) + 1.221)(0.1) = 1.3675 \end{aligned}$$

Hence

$$y(0.3) \approx y_3 = 1.3675$$

3. Find an explicit solution of the following (separable) differential equation.

$$x + \sin(y)\, y' = 2$$

- We first rewriting the ODE in the form $M(x) + N(y)\frac{dy}{dx} = 0$. This requires moving the constant 2 to the left hand side. We then have

$$M(x) = x - 2 \quad , \quad N(y) = \sin(y)$$

Since the ODE is separable, we solve it my solving the equivalent algebraic equation

$$\begin{aligned} \int M(x)\, dx + \int N(y)\, dy &= C \quad \Rightarrow \quad \int (x-2)\, dx + \int \sin(y)\, dy = C \\ &\Rightarrow \quad \frac{1}{2}x^2 - 2x - \cos(y) = C \\ &\Rightarrow \quad \cos(y) = \frac{1}{2}x^2 - 2x - C \\ &\Rightarrow \quad y(x) = \cos^{-1}\left(\frac{1}{2}x^2 - 2x - C\right) \end{aligned}$$

4. Solve the following initial value problem

$$xy' - y = x \quad , \quad y(1) = 0$$

- This is a linear first order ODE. We first put it in the standard form $y' + p(x)\, y = g(x)$, by dividing both sides by x :

$$y' - \frac{1}{x}y = 1 \quad \Rightarrow \quad p(x) = -\frac{1}{x} \quad , \quad g(x) = 1$$

Next, we calculate the integrating factor $\mu(x)$

$$\mu(x) = \exp\left[\int p(x)\, dx\right] = \exp\left[\int -\frac{1}{x} dx\right] = \exp\left[-\ln|x|\right] = x^{-1} = \frac{1}{x}$$

and then

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int \mu(x)\, g(x)\, dx + \frac{C}{\mu(x)} = x \int \frac{1}{x}(1)\, dx + Cx \\ &= x \ln|x| + Cx \end{aligned}$$

Thus, our general solution is $y(x) = \frac{1}{x}\ln|x| + \frac{C}{x}$. We finish by imposing the initial conditions $y(1) = 0$ on the general solution in order to get the appropriate value for the constant C.

$$0 = y(1) = \frac{1}{1}\ln|1| + C(1) = C$$

and so $C = 0$. Our solution is thus

$$y(x) = x \ln|x|$$

5. Consider the following initial value problem

$$\begin{aligned} y + (x+1)\frac{dy}{dx} &= x \\ y(0) &= 1 \end{aligned}$$

Show that the differential equation is exact and then find the explicit solution satisfying the initial condition.

- Putting the ODE in the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$, we find

  $$M(x,y) = y - x \quad , \quad N(x,y) = x + 1$$

  We have

  $$\frac{\partial M}{\partial y} = 1 = \frac{\partial N}{\partial x}$$

  and so the equation **is exact**. This means that the differential equation is derivable from an algebraic equation of the form

  $$\Phi(x,y) = C$$

  with $\Phi(x,y)$ determined by the requirements that $\frac{\partial \Phi}{\partial x} = M(x,y)$ and $\frac{\partial \Phi}{\partial y} = N(x,y)$. Thus,

  $$\begin{aligned} \Phi(x,y) &= \int M(x,y)\,\partial x + c_1(y) = \int (y-x)\,\partial x + c_1(y) = xy - \frac{1}{2}x^2 + c_1(y) \\ \Phi(x,y) &= \int N(x,y)\,\partial y + c_2(x) = \int (x+1)\,\partial y + c_2(x) = xy + y + c_2(x) \end{aligned}$$

  In order to get these two expressions for $\Phi(x,y)$ to agree, we have to choose $c_1(y) = y$ and $c_2(x) = -\frac{1}{2}x^2$. This yields $\Phi(x,y) = xy - \frac{1}{2}x^2 + y$. We now need to solve $\Phi(x,y) = C$, or

  (*) $$xy - \frac{1}{2}x^2 + y = C$$

  But since we have C already isolated on the right hand side, we'll figure out the right value for C before solving (*) for y. The initial condition says that when $x = 0$ we have $y = 1$. Thus, we need

  $$(0)(1) - \frac{1}{2}(0)^2 + (1) = C \qquad \Rightarrow \qquad C = 1$$

  We now solve

  $$xy - \frac{1}{2}x^2 + y = 1$$

  for y. The result is

  $$y = \frac{1 + \frac{1}{2}x^2}{x+1}$$

6. Find the general solution of

(*) $$x^2\frac{dy}{dx} = xy - y^2$$

(Hint: recast the differential equation into the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$ and use the change of variable $z(x) = \frac{y(x)}{x}$.)

- Dividing both sides of the ODE by x^2 yields

$$\frac{dy}{dx} = \frac{y}{x} - \left(\frac{y}{x}\right)^2$$

which is of the desired form with $F(z) = z - z^2$.

Next, we make the suggested change of variable

$$z = \frac{y}{x} \quad \Rightarrow \quad y = zx \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z$$

Substituting z for $\frac{y}{x}$ on the right hand side of (*) and $x\frac{dz}{dx} + z$ for $\frac{dy}{dx}$ on the left hand side of (*), we get

$$x\frac{dz}{dx} + z = z - z^2$$

or, after cancelling the common term on both sides

$$x\frac{dz}{dx} = -z^2$$

or, after dividing both sides by xz^2 and moving terms around,

$$\frac{1}{x} + \frac{1}{z^2}\frac{dz}{dx} = 0$$

This last equation is **separable**. So we solve it by the standard method for separable equations:

$$M(x) = \frac{1}{x} \quad , \quad N(z) = \frac{1}{z^2} \quad \Rightarrow \quad \int \frac{1}{x}dx + \int \frac{1}{z^2}dz = C$$

or

$$\ln|x| - \frac{1}{z} = C$$

or

$$z = \frac{1}{\ln|x| - C}$$

Lastly, we use $y = xz$ to get the solution to the original differential equation.

$$y(x) = \frac{x}{\ln|x| - C}$$