Math 2233
Hint for HW 2 Problem 5

Note: Since MyLab Math randomizes the homework problems for each student, this version of problem 5 may not be exactly what MyLab Math assigned to you.

Problem 2.2.25 in text.

$$x^2\frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)} \qquad , \qquad y(1) = 1$$

- Mulitply both sides of the differential equation by $(y+1)/x^2$ to get

$$(y+1)\frac{dy}{dx} = \frac{4x^2-x-2}{x^2(x+1)}$$

or

$$-\frac{4x^2-x-2}{x^2(x+1)} + (y+1)\frac{dy}{dx} = 0$$

So the differential equation is of the separable form $M(x) + N(y)\frac{dy}{dx} = 0$, with

$$M(x) = -\frac{4x^2-x-2}{x^2(x+1)} \qquad , \qquad N(y) = y+1$$

Therefore, we can solve it by solving instead

$$\int M(x)\,dx + \int N(y)\,dy = C$$

Now

$$\begin{aligned}\int M(x)\,dx &= -\int \frac{4x^2-x-2}{x^2(x+1)}dx \\ &= -\frac{1}{x}\left(3x\ln(x+1) + x\ln x + 2\right)\end{aligned}$$

(This computation is done at the end of this hint.) and

$$\int N(y) = \int (y+1)\,dy = \frac{1}{2}y^2 + y$$

Thus, we have

$$(*) \qquad -\frac{1}{x}\left(3x\ln(x+1) + x\ln x + 2\right) + \frac{1}{2}y^2 + y = C$$

- Now we're at a convenient point to determine the correct value for the constant C. We apply the initial condition $y(1) = 1$ by substitituting $x = 1$, $y = 1$ into (*). This yields

$$-\left(2\ln|2| + 0 + 2\right) + \frac{1}{2} + 1 = C$$

$$C = -\frac{1}{2} - 2\ln|2|$$

Thus, the implicit solution to the initial value problem is

$$(**) \qquad -\frac{1}{x}\left(3x\ln(x+1) + x\ln x + 2\right) + \frac{1}{2}y^2 + y = -\frac{1}{2} - 2\ln|2|$$

- Equation (**) is quadratic in y, and so it can be solved for y using the Quadratic Formula

$$ay^2 + by + c = 0 \quad \Rightarrow \quad y = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

with

$$\begin{aligned} a &= \frac{1}{2} \\ b &= 1 \\ c &= \frac{1}{2} + 2\ln|2| - \frac{1}{x}\left(3x\ln(x+1) + x\ln x + 2\right) \end{aligned}$$

So

$$\begin{aligned} y(x) &= -1 \pm \sqrt{1 - 2\left(\frac{1}{2} + 2\ln|2| - \frac{1}{x}\left(3x\ln(x+1) + x\ln x + 2\right)\right)} \\ &= \sqrt{2}\sqrt{\frac{1}{x}\left(3x\ln(x+1) - 2x\ln 2 + x\ln x + 2\right)} - 1 \end{aligned}$$

My apologies for such a messy computation. It may be that My Lab Math expected you to the implicit solution (**) as your answer.

Computation of $\int M(x)\,dx$

Note that the integrand of $\int M(x)\,dx$ can be written as

$$-\frac{4}{x+1} + \frac{1}{x(x+1)} + \frac{2}{x^2(x+1)}$$

We have

$$\int -\frac{4}{x+1}dx = -4\ln[x+1]$$

by substituting $u = x + 1$.

$$\int \frac{1}{x(x+1)}dx$$

can be integrated using Partial Fractions

$$\begin{aligned} \frac{1}{x(x+1)} &= \frac{A}{x} + \frac{B}{x+1} \\ &\Rightarrow \quad 1 = A(x+1) + Bx \\ x &= 0 \quad \Rightarrow \quad A = 1 \\ x &= -1 \quad \Rightarrow \quad B = -1 \end{aligned}$$

and so

$$\begin{aligned} \int \frac{1}{x(x+1)}dx &= \int \frac{1}{x}dx - \int \frac{1}{x+1}dx \\ &= \ln|x| - \ln|x+1| \end{aligned}$$

$$\int \frac{2}{x^2(x+1)}dx$$

can also be done using Partial Fractions; however, the Partial Fraction Expansion is slightly more complicated because of the x^2 term in the denominator:

$$\frac{2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\begin{aligned}
&\Rightarrow \quad 2 = Ax(x+1) + B(x+1) + Cx^2 \\
x &= 0 \quad \Rightarrow \quad B = 2 \\
x &= -1 \quad \Rightarrow \quad C = 2 \\
x &= 1 \quad \Rightarrow \quad 2 = 2A + (2)(2) + (2) \\
&\Rightarrow \quad A = -2
\end{aligned}$$

and so

$$\begin{aligned}
\int \frac{2}{x^2(x+1)} dx &= \int \left(-\frac{2}{x} + \frac{2}{x^2} + \frac{2}{x+1} \right) dx \\
&= -2\ln|x| - \frac{2}{x} + 2\ln|x+1|
\end{aligned}$$

Thus

$$\begin{aligned}
\int M(x)\, dx &= -4\ln[x+1] \\
&\quad + \ln|x| - \ln|x+1| \\
&\quad -2\ln|x| - \frac{2}{x} + 2\ln|x+1| \\
&= -\frac{1}{x}\left(3x\ln(x+1) + x\ln x + 2\right)
\end{aligned}$$