Math 2233

The Laplace Transform Method for Nonhomogeneous ODEs: Example

$$\begin{aligned} y'' + 8y' + 15y &= 120e^t \\ y(0) &= -7 \\ y'(0) &= 53 \end{aligned}$$

Taking the Laplace transform of the ODE yields

$$\left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 8\left(\mathcal{L}[y] - y(0)\right) + 15\mathcal{L}[y] = 120\mathcal{L}\left[e^t\right]$$

or

$$\left(s^2 + 8s + 15\right)\mathcal{L}[y] + 7s - 53 + 56 = \frac{120}{s-1}$$

or

$$(s+3)(s+5)\mathcal{L}[y] = \frac{120}{s-1} - 7s - 3$$

or

$$\begin{aligned} \mathcal{L}[y] &= \frac{120}{(s-1)(s+3)(s+5)} - \frac{7s+3}{(s+3)(s+5)} \\ &= \frac{-7s^2+4s+123}{(s+5)(s-1)(s+3)} \end{aligned}$$

: Now we have $\mathcal{L}[y]$ expressed as a ratio of two polynomials. Next, we set up a Partial Fractions Expansion:

$$\frac{-7s^2+4s+123}{(s+5)(s-1)(s+3)} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{s+3}$$

We now need to complete the Partial Fractions Expansion to determine the constants A, B, C. After multiplying both sides by $(s+5)(s-1)(s+3)$, we get

$$-7s^2 + 4s + 123 = A(s-1)(s+3) + B(s+5)(s+3) + C(s+5)(s-1)$$

This last equation has to be true for all s. Choosing some special, convenient, values for s, shows

$$\begin{aligned} s = -5 &\Rightarrow -7(25) + 4(-5) + 123 = A(-6)(-2) + B(0) + C(0) \\ &\Rightarrow -72 = 12A \Rightarrow A = 6 \end{aligned}$$

$$\begin{aligned} s = 1 &\Rightarrow -7 + 4 + 123 = A(0) + B(6)(4) + C(0) \\ &\Rightarrow 120 = 24B \Rightarrow B = 5 \end{aligned}$$

$$\begin{aligned} s = -3 &\Rightarrow -7(9) + 4(-3) + 123 = A(0) + B(0) + C(2)(-4) \\ &\Rightarrow 48 = -8C \Rightarrow C = -6 \end{aligned}$$

Applying these values for A, B, C,

$$\begin{aligned} \mathcal{L}[y] &= \frac{-7s^2+4s+123}{(s+5)(s-1)(s+3)} = \frac{A}{s+5} + \frac{B}{s-1} + \frac{C}{s+3} \\ &= 6\frac{1}{s+5} + 5\frac{1}{s-1} - 6\frac{1}{s+3} \\ &= 6\mathcal{L}\left[e^{-5t}\right] + 5\mathcal{L}\left[e^t\right] - 6\mathcal{L}\left[e^{-3t}\right] \\ &= \mathcal{L}\left[6e^{-5t} + 5e^t - 6e^{-3t}\right] \end{aligned}$$

and so

$$y(t) = 6e^{-5t} + 5e^t - 6e^{-3t}$$