LECTURE 27

Series Solutions about Regular Singular Points

Let's now consider the differential equation

$$2x^2y'' - xy' + (1+x)y = 0 \quad . \tag{27.1}$$

This equation evidently has a regular singular point at $x = 0$. We will look for a solution around $x = 0$ by making an ansatz for $y(x)$ by combining our ansatz for power series solutions about regular points with the ansatz we made for Euler type equations. More explicitly, we shall take

$$y(x) = x^r \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+r} \quad . \tag{27.2}$$

We can suppose without loss of generality that $a_0 \neq 0$; i.e., we assume r to be chosen such that the first nonzero term in the series is $a_o x^r$. Plugging (27.2) into (27.1) yields

(27.3)

$$\begin{aligned} 0 &= 2x^2 \textstyle\sum_{n=0}^{\infty} (r+n)(r+n-1)a_n x^{r+n-2} - x \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1} + (1+x)\sum_{n=0}^{\infty} a_n x^{r+n} \\ &= \textstyle\sum_{n=0}^{\infty} 2(r+n)(r+n-1)a_n x^{r+n} - \sum_{n=0}^{\infty}(r+n)a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1} \\ &= \textstyle\sum_{n=0}^{\infty} \left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n x^{r+n} + \sum_{n=1}^{\infty} a_{n-1} x^{r+n} \\ &= (2r)(r-1) - r + 1)\, a_0 + \textstyle\sum_{n=1}^{\infty} \left(\left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n + a_{n-1}\right) x^{r+n} \end{aligned}$$

Hence, we need

$$0 = (2r)(r-1) - r + 1 = 2r^2 - 3r + 1 \tag{27.4}$$

$$0 = a_{n-1} + \left(2(r+n)(r+n-1) - (r+n) + 1\right) a_n \tag{27.5}$$

The first relation is a quadratic equation for r. It is called the **indicial equation** for (27.1). Since

$$2r^2 - 3r + 1 = (2r-1)(r-1) \tag{27.6}$$

we must have

$$r = \frac{1}{2}, 1 \tag{27.7}$$

The second equation (27.5) furnishes a recursion relation that allows us to fix all coefficients a_n in terms of a_o and r.

Setting $r = \frac{1}{2}$ we have

$$\begin{aligned} 0 &= a_{n-1} + \left(2(\tfrac{1}{2}+n)^2 - 3(\tfrac{1}{2}+n) + 1\right) a_n \\ &= a_{n-1} + [n(2n-1)]\, a_n \end{aligned} \tag{27.8}$$

so

$$a_n = \frac{-a_{n-1}}{n(2n-1)} \tag{27.9}$$

Thus,

$$\begin{aligned} a_1 &= \tfrac{-a_0}{(1)(2-1)} = -a_0 \\ a_2 &= \tfrac{-a_1}{(2)(4-1)} = \tfrac{a_0}{6} \\ a_3 &= \tfrac{-a_2}{(3)(6-1)} = \tfrac{-a_0}{90} \end{aligned} \tag{27.10}$$

So one solution would be

$$y_1(x) = a_0 x^{1/2}\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) \quad . \tag{27.11}$$

When $r = 1$ we have

$$0 = a_{n-1} + \left(2(1+n)^2 - 3(1+n) + 1\right) a_n \tag{27.12}$$

or

$$a_n = \frac{-1}{2(1+n)^2 - 3(1+n) + 1} a_{n-1} = \frac{-a_{n-1}}{n(2n+1)} \quad . \tag{27.13}$$

So

$$\begin{aligned} a_1 &= \tfrac{-a_0}{1(2+1)} = -\tfrac{a_0}{3} \\ a_2 &= \tfrac{-a_1}{2(4+1)} = \tfrac{a_0}{30} \\ a_3 &= \tfrac{-a_2}{3(6+1)} = -\tfrac{a_0}{630} \end{aligned} \tag{27.14}$$

Thus, a second solution of (27.1) would be

$$y_2(x) = a_o x\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \cdots\right) \quad . \tag{27.15}$$

The general solution of (27.1) will be a linear combination of $y_1(x)$ and $y_2(x)$:

$$y(x) = c_1 x^{1/2}\left(1 - x + \frac{1}{6}x^2 - \frac{1}{90}x^3 + \cdots\right) + c_2 x\left(1 - \frac{1}{3}x + \frac{1}{30}x^2 - \frac{1}{630}x^3 + \cdots\right) \quad . \tag{27.16}$$

In summary, to find a solution of (27.1), we

1. Assume there is a solution of the form $y(x) = x^r \sum_{n=0}^{\infty} a_n x^n$, with $a_0 \neq 0$.
2. Plug this expression for $y(x)$ into the differential equation and set the total coefficients of each power of x equal to zero. This lead to
   (i) a quadratic equation for r (the indicial equation)
   (ii) a set of recursion relations relating the coefficients a_n
3. Find the two roots r_1 and r_2 of the indicial equations, and then, for each root r_i used the recursion relations to express all the coefficients a_n in terms of a_o.
4. Write down a corresponding solution for each root y_i(x) for each root r_i of the indicial equation.
5. Write down the general solution as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \quad .$$

WARNING: This technique works produces two linearly independent solutions only when:

(i) **There are two distinct roots $\mathbf{r}_1$ and $\mathbf{r}_2$ of the indicial equation.**
(ii) **The difference $\mathbf{r}_1 - \mathbf{r}_2$ is not an integer.**

See Sections 5.7 and 5.8 of the text for a discussion of what happens and how to procede when these criteria are not meet.

Let's do another example

Example 27.1.

$$2xy'' + y' - y = 0$$

This equation has a regular singular point at $x = 0$. So we'll try to find a solution in the form of a generalized power series about $x = 0$.

Ansatz: $y = \sum_{n=0}^{\infty} a_n x^{n+r}$

$$\begin{aligned}
0 &= 2xy'' + y' - y \\
&= \sum_{n=0}^{\infty} 2(n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r)\, a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} \left[(n+r)(2n+2r-2)\, a_n + (n+r)\, a_n\right] x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} \left[(n+r)(2n+2r-1)\, a_n\right] x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=-1}^{\infty} \left[(n+r+1)(2n+2r+1)\, a_{n+1}\right] x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= r(2r-1)\, a_0 x^{r-1} + \sum_{n=0}^{\infty} \left[(n+r+1)(2n+2r+1)\, a_{n+1}\right] x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= r(2r-1)\, a_0 x^{r-1} + \sum_{n=0}^{\infty} \left[(n+r+1)(2n+2r+1)\, a_{n+1} - a_n\right] x^{n+r}
\end{aligned}$$

We now demand that the total coefficient of each distinct power of x vanish. This leads us to the following equations

$$\begin{aligned}
r(2r-1)\, a_0 &= 0 \\
(n+r+1)(2n+2r+1)\, a_{n+1} - a_n &= 0 \quad , \quad n = 0, 1, 2, 3, \ldots
\end{aligned}$$

We always assume that $a_0 \neq 0$ (otherwise the leading term of our ansatz for y is not $a_0 x^r$). Hence, the first equation requires

$$r(2r-1) = 0 \quad \Rightarrow \quad r = 0, \frac{1}{2}$$

We thus have determined that there are two and only two possible choices for r. The coefficients a_n will be determined by

$$a_{n+1} = \frac{a_n}{(n+r+1)(2n+2r+1)} \quad , \quad n = 0, 1, 2, 3, \ldots$$

- Solution with $r = 0$

  The recursion relations in this case reduce to

$$a_{n+1} = \frac{a_n}{(n+1)(2n+1)} \quad , \quad n = 0, 1, 2, 3, \ldots$$

Thus, if $a_0 = c_1$, then

$$\begin{aligned} a_1 &= \frac{a_0}{(1)(1)} = c_1 \\ a_2 &= \frac{a_1}{(2)(3)} = \frac{c_1}{6} \\ a_3 &= \frac{a_2}{(3)(5)} = \frac{c_1}{90} \end{aligned}$$

and so the first four terms of this solution will be

$$\begin{aligned} y &= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \cdots \\ &= c_1 + c_1 x + \frac{c_1}{6} x^2 + \frac{c_1}{90} x^3 + \cdots \\ &= c_1 y_1 \end{aligned}$$

where

$$y_1 = 1 + x + \frac{1}{2} x^2 + \frac{1}{90} x^3 + \cdots$$

- Solution with $r = \frac{1}{2}$

  In this case the recursion relations reduce to

$$a_{n+1} = \frac{a_n}{(n+r+1)(2n+2r+1)} = \frac{a_n}{\left(n+\frac{3}{2}\right)(2n+2)} = \frac{a_n}{(2n+3)(n+1)} \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

  Setting $a_0 = c_2$ we then get

$$\begin{aligned} a_1 &= \frac{a_0}{(3)(1)} = \frac{c_2}{3} \\ a_2 &= \frac{a_1}{(5)(2)} = \frac{c_2}{30} \\ a_3 &= \frac{a_2}{(7)(3)} = \frac{c_2}{630} \end{aligned}$$

and so the first four terms of the solution will be

$$\begin{aligned} y &= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \cdots \\ &= c_2 x^{\frac{1}{2}} + \frac{c_2}{3} x^{3/2} + \frac{c_2}{30} x^{5/2} + \frac{c_2}{30} x^{7/2} + \cdots \\ &= c_2 \sqrt{x} \left(1 + \frac{1}{3} x + \frac{1}{30} x^2 + \frac{1}{630} x^3 + \cdots \right) \\ &= c_2 y_2 \end{aligned}$$

where

$$y_2(x) = \sqrt{x} \left(1 + \frac{1}{3} x + \frac{1}{30} x^2 + \frac{1}{630} x^3 + \cdots \right)$$