LECTURE 1

Introduction

A **differential equation** is an equation stating a relationship between a function, its underlying variables, and one or more of its derivatives. For example, the differential equation

$$\frac{df}{dt} = 3f \tag{1.1}$$

states that the first derivative of a function f with respect to its underlying variable t is always three times the value of f. Such an equation could arise in a purely mathematical context (e.g, f is a function such that the slope of its graph is always three times its value); or as the mathematical expression of a purely empirical phenomenon (e.g., the rate of growth of a colony of bacteria is observed to three times its size. The purpose of this course is to teach you some basic techniques for solving differential equations, to study the general properties of the solutions of differential equations, and to help you develop some intuition with respect to the physical interpretation of differential equations.

Before abstracting things mathematically, it is worthwhile to first make some general comments about why differential equations are so prevalent.

Fundamental to any empirical science is the notion of functions. We use functions to describe (or more accurately, to provide a mathematical model for) how one quantity relates to another. For example, a mechanical engineer might be concerned with how the strength of a construction depends on a design parameter; an ecologist might wish to know the relationship between the populations of various competing species; a chemist might wish to know how reaction rates are affected by the concentrations of reactants. One might think that it would be sufficient to take a lot of data and then guess the form of a function that best fits the curve (say, by adjusting polynomial coefficients); but this approach has several pitfalls:

(1) In general, with each new data point added, the function that replicates the data changes.
(2) Given (1), any prediction made from the guessed function is tenuous (particularly predictions about other experiments or situations).
(3) This method of understanding rarely leads to insights concerning deeper relationships between quantities.

Given the difficulties in making an accurate functional model from a set of data, it is at first surprising that one can actually do better with less (albeit, more profound) information.

Example 1.1. An Italian teenager, out of sheer boredom, starts dropping water balloons off the side of the Tower of Pisa. After hitting a few dozen tourists on the head, he notices that not only do all water balloons fall at the same rate, but that the speed at which they hit the bystanders below is directly proportional to the time it took them to reach their target. As the arresting officer explained, “All objects accelerate towards the ground when they are released, and the magnitude of that acceleration is always the same; 9.8 meters per second per second. That’s the Law.” (The Law of Gravity, of course).

Now acceleration is just the rate of change of velocity with respect to time; so what the officer was saying is that

$$\frac{dv}{dt} = constant = 9.8\ \mathrm{m/sec}^2. \tag{1.2}$$

Integrating both sides and applying the Fundamental Theorem of Calculus (the theorem that says that integration is the reverse of differentiation) to the left hand side we arrive at

$$v = 9.8t.$$

Which tells us precisely the speed of a falling water balloon after t seconds; thus eliminating the need for further experimentation and the potential for a lengthy judicial holiday.

Actually I've been a bit sloppy here, as I have grossly over-simplified the Fundamental Theorem of Calculus. For not only do we have

$$\frac{d}{dt}(9.8t) = 9.8$$

but also

$$\begin{aligned} \frac{d}{dt}(9.8t + 20) &= 9.8 \\ \frac{d}{dt}(9.8 - 32) &= 9.8 \\ & \quad etc. \end{aligned}$$

Thus, the functions

$$\begin{aligned} v_1(t) &= 9.8t \quad , \\ v_2(t) &= 9.8t \quad , \\ v_3(t) &= 9.8t - 32 \quad , \\ & \quad etc \end{aligned}$$

all satisfy the same differential equation.

A bit more precisely, what the Fundamental Theorem really says is that if $f(t)$ is any differentiable function that satisfies

$$\frac{df}{dt} = F(t)$$

then any other function $g(t)$ with the same property (i.e., $\frac{dg}{dt} = F(t)$) must be of the form then it must be of the form

$$g(t) = f(t) + C$$

where C is some constant number (a so-called *constant of integration*). Thus, the most general solution of (1.2) is actually

$$v(t) = 9.8t + C.$$

Thus, in describing mathematically all the solutions to (2), an arbitrary constant C appears. Yet this constant also has a vital physical interpretation: at $t = 0$ we have

$$v(0) = 9.8(0) + C = C$$

In other words, C is the velocity when $t = 0$. Thus, allowing for non-zero values for C allows us to model mathematically situations where the water balloons are thrown down (as opposed to simply being released).

The moral of this example, is that the differential equation that governs this simple experiment not only allows us to replicate the observed velocities for a given situation (in the case at hand, a given initial velocites), but also allows us to extend or model to more general situations.

EXAMPLE 1.2. A cell biologist monitors the growth of a bacteria colony. After taking hundreds of data points, she observes that the colony seems to grow at a rate proportional to its size (which makes sense since the more cells there are in general, the more cells there will be that are undergoing mitosis at a given time). She checks her data and sees that it can be summarized by the following rule

$$\frac{P(t + \Delta t) - P(t)}{\Delta t} = 0.07P(t)$$

where $P(t)$ is the population measured at time t and Δt is the time interval between successive measurements. Lucky for her, she sits next to a calculus student on the bus home. "I hate limits" he says. At that point she recognizes that the left hand side *does* look like the definition of a derivative. So she chucks out her reams of data, and starts with the following differential equation

$$\frac{dP}{dt} = 0.07P.$$

Multiplying both sides by $\left(\frac{dt}{P}\right)$, she obtains

$$\frac{dP}{P} = 0.07dt$$

Integrating both sides then yields

$$\log(P) = 0.07t + C$$

where C is an arbitrary constant of integration. Exponentiating both sides of this equation yields

$$P = \exp(\log(P)) = \exp(0.07t + C) = e^{0.07t+C} = e^{0.07t}e^{C} = P_o e^{0.07t}$$

where $P_o = e^C$ (which is an arbitrary number since C is an arbitrary number).

The biologists then notes that since the original population was 1,078,430, the appropriate value of P_o should be

$$1078430 = P(0) = P_o e^0 = P_o$$

Thus, the general formula for the population of bacteria should be

$$P(t) = 1078430e^{0.07t}$$

Curious, the biologist decides to calculate the present size of the colony, 3 days (=4320 minutes) after the experiment began. After fiddling with her calculator she suddenly bolts out of the lab. Why?