LECTURE 2

Solutions and Classification of Differential Equations

1. Solutions of Ordinary Differential Equations

DEFINITION 2.1. *A **solution** of an (ordinary) differential equation*

$$F\left(x, f, f', f'', \dots, f^{(n)}\right) = 0$$

on the interval $[a,b]$ *is a function* ϕ *such that* $\phi', \phi'', \dots, \phi^{(n)}$ *exist for all* $x \in [a,b]$ *and*

$$F\left(x, \phi(x), \phi'(x), \phi''(x), \dots, \phi^{(n)}(x)\right) = 0$$

for all $x \in [a,b]$.

To verify that a given function $\phi(x)$ is a solution of a differential equation, one uses the rules of differentiation to compute explict expressions for the derivatives $\phi'(x), \phi''(x), \dots$ and then checks that when the derivative terms in the differential equation are replaced by their explicit expressions, the stated identity is true.

EXAMPLE 2.2. A solution of

$$\frac{df}{dx} = -\lambda f$$

on the open interval $(-\infty, +\infty)$ is the function

$$\phi(x) = Ae^{-\lambda x} \quad .$$

- To verify this statement we use the rules of differentiation to calculus to compute the left hand side of the differentiation equation:

  $$\frac{d\phi}{dx} = \frac{d}{dx}\left(Ae^{-\lambda x}\right) = (-\lambda) Ae^{-\lambda x}$$

  Substituting $Ae^{-\lambda x}$ for f and $(-\lambda) Ae^{-\lambda x}$ for $\frac{df}{dx}$in the differential equation yields

  $$(-\lambda) Ae^{-\lambda x} = -\lambda \left(Ae^{-\lambda x}\right)$$

EXAMPLE 2.3. A solution o

$$\frac{d^2 f}{dx^2} + \omega^2 f = 0$$

is

$$\phi_1 = A \sin \omega x \quad .$$

But note that

$$\phi_2 = B \cos \omega x$$

is also a solution.

2. Classification of Differential Equations

The purpose of this course is to teach you some basic techniques for "solving" differential equations and to study the general properties of the solutions of differential equations. I put the word solving in quotes because we will rarely find the solutions of a differential equation by systematically manipulating an equation until the unknown function is isolated. More often we shall find the solutions by first constructing, or even guessing, some solutions and then applying some general theorems to verify that every solution can be found in this manner. Indeed, there exists no general algorithm for solving differential equations. However, if one knows how to construct a solution for one differential equation, one can often generalize the technique to a whole class of differential equations. For this reason the first step in solving a differential equation is to identify its general class.

2.1. Partial vs. Ordinary Differential Equations. An **ordinary differential equation** is an equation relating a function and its derivatives with respect to a single variable.

A **partial differential equation** is one relating a function of more than one variable to its partial derivatives.

Example 2.4.

$$\frac{d\phi}{dx}(x) + g(x)\phi(x) = 0$$

is an ordinary differential equation, while

$$\frac{\partial \Phi}{\partial x}(x,y) + G(x,y)\Phi(x,y) = 0$$

is a partial differential equation.

In this course we shall deal almost exclusively with ordinary differential equations.

2.2. The Order of a Differential Equation. The **order** of a differential equation is the highest number of derivatives appearing in the equation. Thus,

$$2x^2\frac{df}{dx} + \sqrt{x}\frac{d^3f}{dx^3} + e^x = 2$$

is an third order, ordinary, differential equation, while

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial^2\phi}{\partial t^2} = 0$$

is a second order, partial, differential equation.

2.3. Linear vs Non-Linear Differential Equations. A function of a single variable is said to be *linear* if it is of the form $f(x) = Ax + B$ with A and B constants. The nomenclature "linear" comes from the fact that the graph of such a function is always a straight line (with slope A and y-intercept B). For the purpose of generalizing this property, let me introduce an equivalent condition.

Definition 2.5. *A funciton of one variable is* ***linear*** *if its second derivative is always equal to* 0.

In other words, a function $f(x)$ is linear if it satisfies

$$\frac{d^2f}{dx^2}(x) = 0 \qquad \text{for all } x$$

This latter condition allows a simple extension to more than one variable:

DEFINITION 2.6. *A function F of several variables $x_1, x_2, x_3, \ldots$ is **linear** with respect to $x_1, x_2, x_3, \ldots$ if all its second derivatives $\frac{\partial^2 F}{\partial x_{ii} \partial x_j}$, $i, j = 1, 2, 3, \ldots$, are equal to 0 (thus, in particular, $\frac{\partial^2 F}{\partial x^2} = 0$, $\frac{\partial^2 F}{\partial x \partial y} = 0$, etc.)*

DEFINITION 2.7. *An ordinary or partial differential equation is said to be **linear** if it is linear in the "unknown function" and its derivatives.*

Thus, a general, linear, ordinary, n^{th} order, differential equation would be one of the form

$$a_n(x) \frac{d^n f}{dx^n}(x) + a_{n-1}(x) \frac{d^{n-1} f}{dx^{n-1}}(x) + \cdots + a_1(x) \frac{df}{dx}(x) + f(x) = g(x) \quad .$$

It is important to note that the functions $a_n(x), \ldots, a_1(x), g(x)$ need not be linear functions of x. The following two examples should convey the general idea.

EXAMPLE 2.8.

$$x^2 \frac{\partial f}{\partial x} + z \frac{\partial^2 f}{\partial y^2} = e^{zxy}$$

is a 2^{nd} order, linear, partial, differential equation.

EXAMPLE 2.9.

$$\frac{d^3 f}{dx^3} + x^2 \frac{df}{dx} + f^2 = 1$$

is a non-linear, ordinary, differential equation of order 3. The equation is non-linear arises because of the presence of the term f^2 which is a quadratic function of the unknown function f.

3. Systems of Differential Equations

A system of algebraic equations is simply a set of equations

$$\begin{aligned} a_1 x + a_2 y + \cdots + a_n z &= a_0 \\ b_1 x + b_2 y + \cdots + b_n z &= b_0 \\ &\vdots \\ s_1 x + s_2 y + \cdots + s_n z &= s_0 \end{aligned}$$

which are all presumed to hold simultaneously.

Similarly, one can consider systems of differential equations; for example,

$$\begin{aligned} f'' + f'g + x^2 g &= 0 \\ g' + x^3 &= h'' \\ \sqrt{x} f' + h' - x^2 g &= 0 \end{aligned}$$

and look for a set of functions f,g,h which satisfy all these equations simultaneously. Such systems arise, for example, when studying chemical reactions where the rate at which a given reactant is consumed is dependent on the concentration of other other reactants in the chemical process.