LECTURE 3

Graphical and Numerical Methods

Consider the graph of a solution $x(t)$ of the differential equation

$$\frac{dx}{dt} = F\left(x(t), t\right) \tag{3.1}$$

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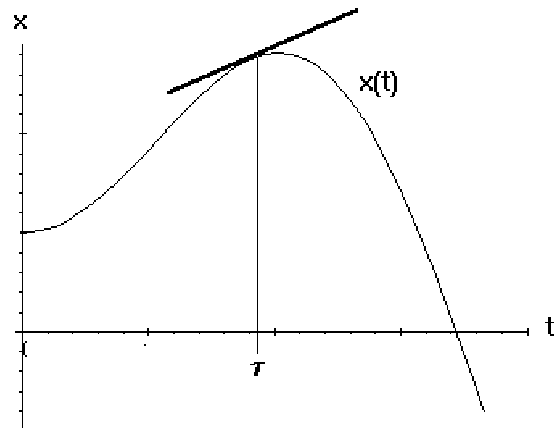
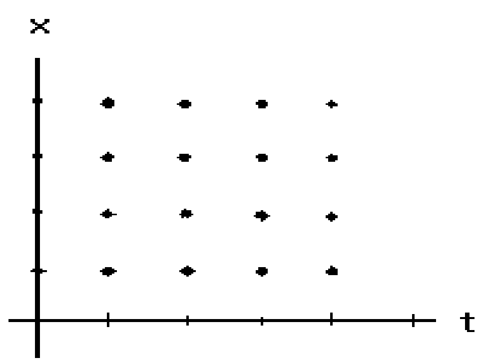
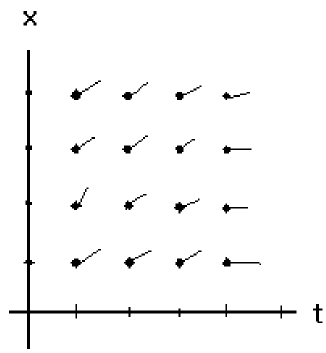


Now $\frac{dx}{dt}(\tau)$ is precisely the slope of the graph of $x(t)$ at the point $(\tau, x(\tau))$. Thus, since $x(t)$ is to be a solution of the differential equation (3.1), we can conclude the that slope of the graph of $x(t)$ at the point $(\tau, x(\tau))$ is exactly $F\left(x(\tau), \tau\right)$.

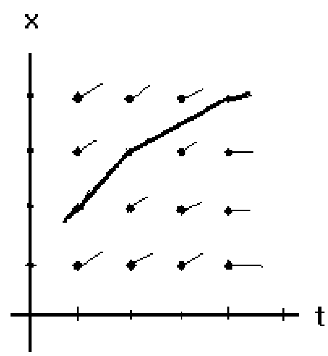
Now let's remove the graph of $x(t)$ from the picture, and look instead a grid of points in the tx-plane:



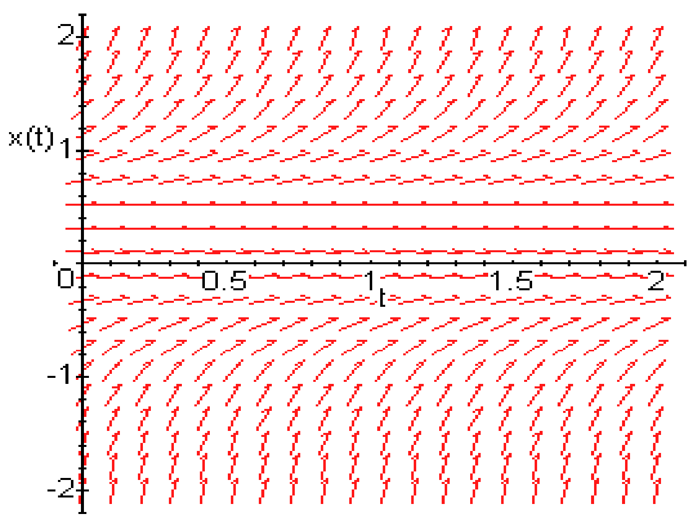
We still know that the slope of the solution that passes thru the point (t, x) must be given by $F(x, t)$.. Therefore, to get a picture of the possible solutions of the differential equation (3.1) we can pick a bunch of sample points (t_i, x_j) forming a nice rectangular grid in the tx-plane, calculate the value of $F(x,t)$ at each of these points, and then draw short lines with slopes $F(x_j, t_i)$ passing through the points $(t_i x_j)$



and then finally we can try to draw curves that pass thru all the points (t_i, x_j) in such a way that their tangent lines are always parallel to the lines eminating from each of the points (t_i, x_j).



If you do this for a large number of points you can get a fairly accurate picture of a large number of solutions of your differential equation.



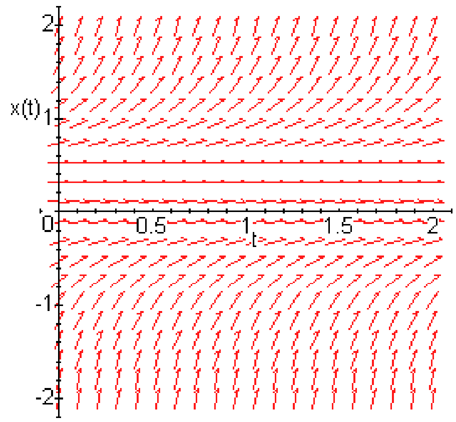


Figure 1

The graph above corresponds to the differential equation

$$\frac{dx}{dt} = t\sin(x).$$

It was produced by Maple via the following commands:

(1) with(DEtools);
(2) dfieldplot(diff(x(t),t) = t*sin(x),[x],t=0..2,x=0..2);

0.1. Interpretation of Graphical Solutions. What's nice about the graphical method described above is that it gives a fairly accurate view of *all* solutions (in a given region of the tx-plane) of a first order differential equation. Of course accuracy here does not mean numerical accuracy. What I mean to say is that the picture itself is enough to provide accurate knowledge about the solutions.

Example 3.1. Sketch the direction fields associated with the following differential equation

$$\dot{x} = x(x-1)$$

Below is the output of the Maple command "dfieldplot(diff(x(t),t) = x*(2*x -1),[x],t=0..2,x=-2..2);":

Example 3.2. Now suppose this differential equation describes the position of a particle as a function of time. Can you make any predictions about the trajectories of particles as $t \to \infty$?

Let's look at the direction field plot. Note that at all points above the line $x = 1$, the direction field vectors have positive slope. This means the the solutions which have at least one point above the line $x = 1$ are always increasing (their tangent vectors always have positive slope). So any solution $x(t)$ that starts off above the line $x = 1$ will tend to infinity as t goes to infinity.

What about solutions that pass through the line $y = 1$? Well, the direction field vectors are identically zero along the line $x = 1$. So the slope of any solution $x(t)$ passing through the line $y = 1$ is constant and equal to zero. Therefore, once a solution reaches the line $x = 1$, it stays there.

At this point, it might be helpful to look specifically at the sign of the function $F(x,t) = x(x-1)$ that defines the differential equation in the various regions of the xt-plane:

Region	$sign(\frac{dx}{dt}) = sign(F(x,t))$
$x > 1$	positive
$x = 1$	zero
$0 < x < 1$	negative
$x = 0$	zero
$x < 0$	positive

Thus, if a solution starts off in the region $x > 1$ then its slope is always positive, and so such a solution would tend to ∞ as $t \to \infty$.

If a solution starts off with $x = 1$, then its slope is initially zero, and so the function is initially constant. But then it can never leave the line $x = 1$. And so such a solution will just be the constant solution $x(t) = 1$

If a solution starts off with $0 < x < 1$, then its slope is initially negative, so the function is initially decreasing. However, at $x = 0$, the slope is zero again, so the solution cannot decrease any further. Such solutions will thus asymptotically approach the line $x = 0$ ast $t \to \infty$.

If a solution starts off with $x = 0$, then the slope is initially zero and remains at zero. Thus, such a solution will always be the constant solution $x(t) = 0$

If a solution starts off with $x < 0$, then its slope will be initially positive. However, such a solution can not increase past the value $x = 0$ since the slope must be zero along the line $x = 0$. Therefore, such a solution will asymptotically approach the line $x = 0$ as $t \to \infty$.

1. Euler's Method

As discussed above, direction field plots provide a nice visual way of interpreting first order ODEs and their solutions. However, they're not all that precise in providing the values of a particular solution of a differential equation. They just good up to the "eyeball" technique by which one sketches the solution.

There is another, more algorithmic, method that can be used to find good approximations to the actual values $y(x)$ of a solution of a differential equation satisfying a given initial condition

$$\begin{aligned} y' &= F(x,y) \\ y(x_0) &= y_0 \end{aligned}$$

and it is based on basically the same principle as the graphical method discussed above.

The differential equation says that if a solution passes thru the point (x_0, y_0) in the xy-plane, then it does so with slope $F(x_0, y_0)$. This then gives us a straight line that best fits the actual solution near (x_0, y_0). If we follow that line out a bit by increasing x_0 to $x_0 + \Delta x$, then along that line y changes by

$$\Delta y = (slope)\,\Delta x = F(x_0, y_0)$$

(recall that the slope of a line is defined as $\frac{\Delta y}{\Delta x}$). Thus, along that line that best fits the solution near (x_0, y_0) if $x \to x_0 + \Delta x$, $y \to y_0 + \Delta y = y_0 + F(x_0, y_0)\,\Delta x$. Thus, if we set

$$\begin{aligned} x_1 &= x_0 + \Delta x \\ y_1 &= y_0 + F(x_0, y_0)\,\Delta x \end{aligned}$$

then the point (x_1, y_1) should be a good approximation to a point on the actual solution.

But now we can repeat the process, assuming (x_1, y_1) is on the solution, we can compute the slope of the solution there as $F(x_1, y_1)$, find a new line that best approximates the actual solution $y(x)$ near the point

(x_1, y_1) and then use that line to identify a third approximate point on the solution curve

$$\begin{aligned} x_2 &= x_1 + \Delta x \\ y_2 &= y_1 + F(x_1, y_1)\,\Delta x \end{aligned}$$

Repeating this process over and over we can readily generate a table of approximate values for the solution of the stated differerential equation and initial condition.

EXAMPLE 3.3. Consider the following initial value problem.

$$\begin{aligned} y' &= x^2 y \\ y(1) &= 2 \end{aligned}$$

Estimate $y(1.3)$ using step sizes of $\Delta x = 0.1$.

The initial condition gives an initial value, 1, for x and an initial value, 2, for y. Thus we set,

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 2 \end{aligned}$$

Now if we increase x to

$$x_1 = x_0 + \Delta x = 1 + 0.1 = 1.1$$

and try the follow the best straight line fit to the actual solution through $(1, 2)$, we'll arrive at

$$y_1 = y_0 + F(x_0, y_0)\,\Delta x = 2 + x^2 y\big|_{\substack{x=1 \\ y=2}}\,\Delta x = 2 + (1)^2 (2)(0.1) = 2.2$$

Thus,

$$y(1.1) \approx 2.2$$

Now increase x again to $x_2 = x_1 + \Delta x = 1.2$. The corresponding value of y will be

$$y_2 = y_1 + F(x_1, y_1)\,\Delta x = 2.2 + (1.1)^2 (2.2)(0.1) = 2.4662$$

So

$$y(1.2) \approx 2.4662$$

Repeating this process one more time, we get

$$\begin{aligned} x_3 &= x_2 + \Delta x = 1.3 \\ y_3 &= y_2 + F(x_2, y_2)\,\Delta x = 2.4662 + (1.2)^2 (2.4662)(0.1) = 2.8213 \end{aligned}$$

and we can conclude

$$y(1.3) \approx 2.8213$$

Here is the summary of this numerical method solving initial value probles for first order ODEs.

Given

$$\begin{aligned} y' &= F(x, y) \\ y(x_0) &= y_0 \end{aligned}$$

Choose a (small) Δx and set

$$\begin{aligned} x_1 &= x_0 + \Delta x \\ y_1 &= y_0 + F(x_0, y_0)\,\Delta x \end{aligned}$$

and then compute successive pairs (x_i, y_i) via

$$\begin{aligned} x_i &= x_{i-1} + \Delta x \\ y_i &= y_{i-1} + F(x_{i-1}, y_{i-1}) \end{aligned}$$

Each time you do this you get a new approximate point on the solution of the initial value problem. This will allow you to build up a table of (approximate) values for the actual solution.