LECTURE 4

First Order Differential Equations and the Fundamental Theorem of Calculus

We'll now begin to develop techniques for solving first order differential equations. The general problem is this: given a first order differential equation

$$\frac{dy}{dx} = F(x,y) \tag{1}$$

Find a function $\phi(x)$ such that ϕ is substituted for y on both sides of (1) the resulting equation is a mathematical identity. Unfortunately, we not solve equation (1) in the generality stated. Rather we will have to proceed special case by special case. In the end, however, we'll end up with a substantial repetoire of techniques that will enable us to solve all but the most pathological differential equations of the form (1).

1. The Fundamental Theorem of Calculus

The simplest special case of equation (1) is when the right hand side is a function of x alone. So let us consider differential equations of the form

$$\frac{dy}{dx} = f(x) \tag{2}$$

Solving this equation is equivalent to answering the question: *what function $y(x)$ has the function $f(x)$ as its derivative?* Addressing this question was actually a large part of Calculus I and II. In the contex of Calculus I and II, it was phrased *what is the anti-derivative of* $f(x)$? The answer for us will be essentially the same as the answer in Calculus: we find solutions to (1) (i.e. find the anti-derivative of $f(x)$) by integrating both sides. However, there are some additional nuances and details to be filled in.

This I shall do below.To write down the solution of this equation, it suffices to simply apply the Fundamental Theorem of Calculus. Now roughly speaking the Fundamental Theorem of Calculus says that integrals and derivatives are inverses of each other. Here is a more precise statement

THEOREM 4.1 (Fundamental Theorem of Calculus). *Let f be a continuous function on $[a,b]$.*

I. *For $x \in [a,b]$, let*

$$F_a(x) = \int_a^x f(x)\,dx \equiv \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\,\Delta x \quad , \quad x_i = x_i + i\Delta x \quad , \quad \Delta x = \frac{x-a}{N} \quad . \tag{3}$$

Then $F(x)$ is continuous and differentiable on (a,b) and

$$\frac{dF_a}{dx}(x) \equiv \lim_{\varepsilon\to 0} \frac{F_a(x+\varepsilon) - F_a(x)}{\varepsilon} = f(x)$$

II. *If $F(x)$ is any anti-derivative of $f(x)$ (i.e., any differentiable function whose derivative is equal to $f(x)$) , then*

$$\int_a^x f(x)\,dx = F(x) - F(a) \tag{4}$$

1.1. Digression: Definite Integrals, Indefinite Integrals, and Constants of Integration. What makes the Fundamental Theorem so fundamental is that it connects several different mathematical limit constructions (derivatives, integrals, and anti-derivatives). Unfortunately, several centuries of mathematics has managed to confuse the latter two mathematical constructs by employing similiar notations.

A *definite integral*, on the one hand, is an expression of the form

$$\int_a^b f(x)\,dx$$

where one is to *integrate* a function $f(x)$ along a specific interval $[a,b]$; this will be some number (obtained by taking the limit of a sequence of Riemann sums). Its value is a number, corresponding to the difference between the areas of the graph of $f(x)$ above and below the x-axis, between $x=a$ and $x=b$.

Distinct from this is the notion of an *indefinite integral*

$$\int f(x)\,dx$$

the value of which is to be a function which is the anti-derivative of $f(x)$ (i.e., a function whose derivative is $f(x)$).

If we set

$$F(x) \equiv \int f(x)\,dx \qquad \text{(meaning } F(x) \text{ is an anti-derivative of } f(x)\text{)}$$

then second part of the Fundamental Theorem of Calculus says $\int_a^b f(x)\,dx$ can be computed by taking the difference in the values of $F(x)$ at $x=b$ and $x=a$.

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

(as opposed to computing it via the limit definition on the right hand side of (3)).

Or, in yet another way, the second part of the Fundamental Theorem tells us

$$y(x) \equiv \int_a^x f(x)\,dx = F(x) - F(a)$$

will satisfy

$$\frac{dy}{dx} = f(x) \quad . \tag{5}$$

Let me summarize this discussion by stating a corollary to Fundamental Theorem of Calculus suitable for the problem of solving differential equations like (5).

COROLLARY 4.2. *Any function of the form*

$$y(x) = \int f(x)\,dx + C \qquad (\textit{where } C \textit{ is a constant}) \tag{6}$$

will provide a solution of

$$\frac{dy}{dx} = f(x) \quad .$$

In fact, not only can we construct lots of solutions by adjusing the constant C on the right hand side of (7). We shall see later (Lecture 8) that **every** solution of (5) can be expressed in the form (6). We might as well state this latter result as a theorem (to be proved later):

THEOREM 4.3. *The general solution to*

$$\frac{dy}{dx} = f(x)$$

is given by

$$y(x) = \int f(x)\,dx + C$$

where C is some constant.

By the way, in stating the theorem I tacitly introduced the notion of a *general solution*. This just means a formula that encompasses **all** the solutions. The constant C on the right hand side of (3) is often referred to as either an *artbitary constant* or as a *constant of integration*. Fixing C to be a particular number yields a *particular solution*, and leaving it arbitrary furnishes the *general solution*.

2. Examples

EXAMPLE 4.4. Solve

$$\frac{dy}{dx} = x + \cos(x)$$

- By the corollary

$$\begin{aligned} y(x) &= \int (x + \cos(x))\,dx + C \\ &= \frac{1}{2}x^2 - \sin(x) + C \end{aligned}$$

EXAMPLE 4.5. Solve

$$\frac{dy}{dx} = x\sin(x)$$

- (The main point of this example is to remind you of integration by parts). By the corollary to the Fundamental Theorem, we have

$$y(x) = \int x\sin(x)\,dx + C$$

The Integration by Parts formula says

$$\int udv = uv - \int vdu$$

If we take

$$\begin{aligned} u &= x \quad \Rightarrow \quad du = dx \\ dv &= \sin(x)\,dx \quad \Rightarrow \quad v = \int dv = \int \sin(x)\,dx = -\cos(x) \end{aligned}$$

So

$$\int x\sin(x)\,dx = (x(-\cos(x))) - \int(-\cos(x))\,dx = -x\cos(x) + \int \cos(x) = -x\cos(x) + \sin(x)$$

Thus,

$$y(x) = -x\cos(x) + \sin(x) + C$$

will be the solution to the differential equation.