LECTURE 9

Applications of First Order ODEs

1. Mixing Problems

Consider the following situation. *A 100 gallon tank is initially full of fresh water. It is then flushed out with a salt solution coming in at a concentration of 1/2 lb/gal and at a rate of 2 gal/min. Simultaneously, the well-mixed solution is drained from the tank at a rate of 2 gal/min. Find the concentration of the salt solution as a function of time t.*

Let me denote by $C(t)$ and $Q(t)$, respectively, the concentration of salt at time t and the quantity of salt in solution at time t. Note that

$$\text{concentration} \quad \equiv \quad \frac{\text{quantity}}{\text{volume}}$$

and so, in our situation

$$C(t) = \frac{Q(t)}{100\ gal} \qquad \Longleftrightarrow \qquad Q(t) = 100C(t)$$

The reason for introducting $Q(t)$ is because it is pretty straight-forward to see how the quantity of salt in the tank is changing. The rate at which salt comes into the tank will be equal to the incoming concentration times the rate of flow:

$$\left(0.5\frac{lb}{gal}\right)\left(2\frac{gal}{\text{min}}\right) = 1\frac{lb}{\text{min}}$$

and so, the rate at which salt is coming into the tank is

$$\left(\frac{dQ}{dt}\right)_{in} = 1\frac{lb}{\text{min}}$$

On the other hand, salt is leaving the tank will be equal to the concentration inside the tank times the rate of flow out

$$\left(\frac{dQ}{dt}\right)_{out} = C(t)\left(2\frac{gal}{\text{min}}\right)$$

The total rate of change of Q, will be equal to the difference between the rate at which it comes in and the rate at which it leaves. Thus,

$$\frac{dQ}{dt} = \left(\frac{dQ}{dt}\right)_{in} - \left(\frac{dQ}{dt}\right)_{out} = 1 - 2C(t)$$

(above and oftern until the end, I'll suppress units of measurement). We also have

$$\frac{dQ}{dt} = \frac{d}{dt}(100C(t)) = 100\frac{dC}{dt}$$

and so we arrive at

$$100\frac{dC}{dt} = 1 - 2C(t)$$

or

$$\frac{dC}{dt} + \frac{1}{50}C(t) = \frac{1}{100}$$

Note that this is a first order linear differential equation.

We also have an initital condition. Since the tank is initially filled with fresh water we have

$$C(0) = 0\frac{lb}{gal} \quad .$$

Thus, we have an initial value problem

$$\begin{aligned} \frac{dC}{dt} + \frac{1}{50}C &= \frac{1}{100} \\ C(0) &= 0 \end{aligned}$$

Let's first find the general solution of the differential equation. Since the differential is linear and in standard form, we can readily employ the formulas

$$y' + p(x)\,y = g(x) \quad \Rightarrow \quad y(x) = \frac{1}{\mu(x)}\int \mu(x)\,g(x)\,dx + \frac{C}{\mu(x)} \quad \text{w/} \quad \mu(x) = \exp\left(\int p(x)\right)dx$$

except that we'll use

$$\begin{aligned} x &\rightarrow t \\ y &\rightarrow C \\ p &\rightarrow \frac{1}{50} \\ g &\rightarrow \frac{1}{100} \\ C &\rightarrow \widetilde{C} \quad \text{(relabeled so that we don't confuse the constant of integration with } C(t)) \end{aligned}$$

Thus, we compute

$$\mu(t) = \exp\left(\int \frac{1}{50}dt\right) = e^{t/50}$$

and

$$\begin{aligned} C(t) &= \frac{1}{e^{t/50}}\int e^{t/50}\left(\frac{1}{100}\right)dt \quad + \quad \frac{\widetilde{C}}{e^{t/50}} \\ &= e^{-t/50}\left(\frac{1}{100}\frac{1}{\frac{1}{50}}e^{t/50}\right) \quad + \quad \widetilde{C}e^{-t/50} \\ &= \frac{1}{2} \quad + \quad \widetilde{C}e^{-t/50} \end{aligned}$$

This our general solution. We now fix the constant of integration $\widetilde{C}$ by imposing the initial condition $C(0) = 0$:

$$0 = C(0) = \frac{1}{2} + \widetilde{C}e^{0} = \frac{1}{2} + \widetilde{C} \quad \Rightarrow \quad \widetilde{C} = -\frac{1}{2}$$

This our solution is

$$C(t) = \frac{1}{2} - \frac{1}{2}e^{-t/50} \quad .$$

2. Cooling

Newton's Law of Cooling says that the temperature of an object changes at a rate proportional to the difference between its temperature and the temperature of its surroundings. Consider a cup of Macdonald's coffee. If the coffee had a temperature of $200^\circ F$ when it was poured and 1 minute later it had a temperature of $190^\circ F$, determine its temperature as an explicit function of t.

Well, formulated mathematically, Newton's Law of Cooling is

$$\frac{dT}{dt} = -k\,(T - T_s)$$

Here T represents the temperature of the coffee aqnd the minus sign just because we expect the coffee's temperature to be decreasing if $T_s < T$. This is a first order linear ODE. Putting it in standard form we have

$$T' + kT = kT_s$$

This differential equation is readily solved using the method for linear first order ODEs:

$$\mu(t) = \exp\left(\int k dt\right) = e^{kt}$$

$$\begin{aligned} T(t) &= \frac{1}{e^{kt}} \int e^{kt}(kT_s) + \frac{C}{e^{kt}} \\ &= e^{-kt}\left(\frac{1}{k}e^{kt}\right) kT_s + Ce^{-kt} \end{aligned}$$

or

$$T(t) = T_s + Ce^{-kt}$$

Initially, the temperature is $200°F$ and $T_s = 70°F$, say.

$$200 = T(0) = 70 + Ce^0 \quad \Rightarrow \quad C = 130$$

Thus,

$$T(t) = 70 + 130e^{-kt}$$

We haven't yet figured out the appropriate value for the constant k; but we also have an additional condition in the problem statement. After 1 minute the temperature is $190°F$. Thus,

$$190 = T(1) = 70 + 130e^{-k} \quad \Rightarrow \quad e^{-k} = 120/130$$

or

$$k = -\ln\left(\frac{120}{130}\right) = 0.080$$

Thus,

$$T(t) = 70 + 130e^{-(0.08)t}$$

3. Projectile with Friction Included

Consider a canonball of mass m fired straight up with an intial velocity v_0. Near the surface of the Earth (as we shall assume this entire scenario takes place) the force of gravity is

$$F_g = -mg$$

(the minus sign indicating the gravitational force is in the downward direction). If there is no friction, then F_g is the only force acting on the canonball, and so Newton's Second Law of Motion says

$$ma = m\frac{dv}{dt} = -mg \quad \Rightarrow \quad \frac{dv}{dt} = -g \quad \Rightarrow \quad v(t) = \int (-g)\,dt + C \quad \Rightarrow \quad v(t) = -gt + C$$

The constant of integration C can be determined from the initial condition $v(0) = v_0$:

$$v_0 = v(0) = -g(0) + C = C$$

Thus,

$$v(t) = -gt + v_0$$

Now suppose we include a frictional force of the form

$$F_f = -k|v|$$

(the minus sign now corresponds to the fact that the frictional force will cause a deceleration of the canonball). Newton's Second Law now says

$$ma = \frac{dv}{dt} = -mg - kv$$

or

$$\frac{dv}{dt}+\frac{k}{m}|v|=-g$$

Notice that this differential equation is a first order linear differential equation.

Let's restrict our attention to the upward flight of the canonball (so that $|v|=v$). We then have

$$(*)\qquad \frac{dv}{dt}+\frac{k}{m}v=-g$$

Being of the form

$$y'+P(x)\,y=G(x)$$

which we know has solutions of the form

$$y(x)=\frac{1}{\mu(x)}\int\mu(x)\,G(x)\,dx+\frac{C}{\mu(x)}\qquad\text{where }\mu(x)=\exp\left(\int P(x)\,dx\right)$$

we can rapidly compute the general solution to (*) by substituting

$$x\to t\quad,\quad y\to v\quad,\quad P(x)\to\frac{k}{m}\quad,\quad G(x)\to -g$$

Thus,

$$\mu(t)=\exp\left(\int\frac{k}{m}dt\right)=e^{\frac{k}{m}t}$$

and

$$\begin{aligned}v(t)&=\frac{1}{e^{\frac{k}{m}t}}\int e^{\frac{k}{m}t}(-g)\,dt+\frac{C}{e^{\frac{k}{m}t}}=e^{-\frac{k}{m}t}\left(-g\frac{m}{k}e^{\frac{k}{m}t}\right)+Ce^{-\frac{k}{m}t}\\&=-\frac{gm}{k}+Ce^{-\frac{k}{m}t}\end{aligned}$$

Imposing the initial condition on this solution will give us the appropriate value for C

$$v_0=v(0)=\frac{-gm}{k}+Ce^0\quad\Rightarrow\quad C=v_0+\frac{gm}{k}$$

and so

$$v(t)=-\frac{gm}{k}+\left(v_0+\frac{gm}{k}\right)e^{-\frac{k}{m}t}$$

Does this answer make sense? The functional form of this solution certainly looks very different from the simple linear dependence on t that twe had in the no friction case.

Let's look what happens in a small frictionsituation. The exponential function $e^{-\frac{k}{m}t}$ has as its Taylor expanstion

$$e^{-\frac{k}{m}t}\approx 1+\left(\frac{-k}{m}t\right)+\frac{1}{2}\left(\frac{-k}{m}t\right)^2+\cdots+\frac{1}{n!}\left(-\frac{k}{m}t\right)^n+\cdots$$

Keeping only the linear term (by assuming that $\frac{k}{m}t<<1$), we have

$$\begin{aligned}v(t)&=-\frac{gm}{k}+\left(v_0+\frac{gm}{k}\right)\left(1-\frac{k}{m}t+\cdots\right)\\&=v_0-gt\end{aligned}$$

and so we do recover the no friction case in the limit $k\to 0$.

4. Escape Velocity

Normally, if you fire a projectile out of a cannon, the projectile eventually falls back to Earth. However, if the projectile has a large enough initial velocity, it can escape the Earth's gravity and travel off into infinite space. The initial velocity at which a projectile just manages to escape the Earth's gravitation pull is called the Earth's *escape velocity*.In this application, we'll calculate the Earth's escape velocity.

Newton's Law of Gravity (with his Second Law of Motion) is

$$m\frac{dv}{dt} = -\frac{GmM_E}{r^2} \tag{1}$$

Here $\frac{dv}{dt}$ is the acceleration, m is the mass of the projectile, M_E is the mass of the eartch, r is the distance from between the projectile and the center of the Earth, and G is the universal gravitational constant. For terrestrial problems, one normally set the gravitational force as

$$F = -mg$$

where $g = 32ft/\sec^2.$. But if R_E is the radius of the Earth, we have at the Earth's surface,

$$-mg \approx F = -m\frac{GM_E}{R_E^2} \quad \Rightarrow \quad GM_E = gR_E^2$$

Setting h to be the height above the Earth's surface, we have

$$r = R_E + h$$

and we can refomulate (1) as

$$m\frac{dv}{dt} = -\frac{mgR_E^2}{(R_E+h)^2}$$

or

$$\frac{dv}{dt} = -\frac{gR_E^2}{(R_E+h)^2}$$

At present, we are working with three variables: v, h and t. However, we can use

$$\frac{dv}{dt} = \frac{dv}{dh}\frac{dh}{dt} = v\frac{dv}{dh}$$

to eliminate the time parameter from the problem. Thus, we have

$$v\frac{dv}{dh} = -\frac{gR_E^2}{(R_E+h)^2}$$

This is a Separable First Order ODE for v as a function of h. I'll solve via the mneumonic method

$$\begin{aligned} v\frac{dv}{dh} &= -\frac{gR_E^2}{(R_E+h)^2} \quad \Rightarrow \quad vdv = -\frac{gR_E^2}{(R_E+h^2)}dh \\ &\Rightarrow \quad \int vdv = \int -\frac{gR_E^2}{(R_E+h^2)}dh \quad + \quad C \\ &\Rightarrow \quad \frac{1}{2}v^2 = +\frac{gR_E^2}{(R_E+h)} \quad + \quad C \end{aligned}$$

or

$$v(h) = \pm\sqrt{\frac{2gR_E^2}{R_E+h} + 2C}$$

We also have an initial condition: when the projectile leaves the cannon at $h = 0$, it should have an inital velocity v_0. Thus,

$$v_0 = v(0) = \pm\sqrt{\frac{2gR_E^2}{R_E+0} + 2C} \quad \Rightarrow \quad C = \frac{1}{2}v_0^2 - gR_E$$

Note that we should also take the solution with the + sign in (2) since we only want solutions with positive (upwards) velocity. Thus, we have

$$v(h) = \sqrt{\frac{2gR_E^2}{R_E+h} + v_o^2 - 2gR_E}$$

To find the maximum height that the projectile reaches we set $v(h_{\max}) = 0$ and solve for $h_{\max}$

$$v(h_{\max}) = 0 \quad \Rightarrow \quad \frac{2gR_E^2}{R_E + h_{\max}} = 2gR_E - v_0^2$$

or

$$h_{\max} = \frac{v_0^2 R_E}{2gR_E - v_0^2}$$

Or, inverting this relation, we can find the initial velocity needed to reach a given height $h_{\max}$

$$v_0(h_{\max}) = \sqrt{2gR_E \frac{h_{\max}}{R + h_{\max}}}$$

Finally, letting $h_{\max} \to \infty$, we obtain the escape velocity

$$v_{esape} = \lim_{h_{\max}\to\infty} v_0(h_{\max}) = \lim_{h_{\max}\to\infty} \sqrt{2gR_E \frac{h_{\max}}{R + h_{\max}}} = \sqrt{2gR_E} = 6.9\frac{mi}{\text{sec}} = 11.1\frac{m}{\text{sec}}$$