LECTURE 10

Sample First Exam

1. Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) $y'' + \cos(y) = x$
• 2^{nd} order, nonlinear, ODE

(b) $\dfrac{\partial \Phi}{\partial y} + \dfrac{\partial^2 \phi}{\partial x^2} = y^2$
• 2^{nd} order, linear, PDE

(c) $\dfrac{d^3x}{dt^3} + x^2\dfrac{dx}{dt} + x = 0$
• 3^{rd} order, non-linear, ODE

(d) $a\,(x)\,y' + b\,(x)\,y + c\,(x) = 0$
• 1^{st} order, linear, ODE

(e) $\dfrac{dx}{dt} = x^2$
• 1^{st} order, nonlinear, ODE

2. Consider the following first order ODE: $y' = x + y$ and suppose $y(x)$ is the solution satisfying $y(1) = 1$. Use the numerical (Euler) method with $n = 3$ and $\Delta x = 0.1$ to estimate $y(1.3)$.

• We will begin constructing a table of approximate values for points $(x_i, y_i \approx y\,(x_i))$ on the solution using the Euler formula

$$\begin{aligned} x_{i+1} &= x_i + \Delta x \\ y_{i+1} &= y_i + F\,(x_i, y_i)\,\Delta x \end{aligned}$$

with $F(x, y) = x + y$ and $x_0 = 1$, $y_0 = 1$.

$$\begin{array}{ll} x_1 = x_0 + \Delta x = 1.1 & y_1 = y_0 + m\,(x_0, y_0)\,\Delta x = y_0 + (x_0 + y_0)\,\Delta x = 1 + (1 + 1)\,(0.1) = 1.2 \\ x_2 = x_1 + \Delta x = 1.2 & y_2 = y_1 + m\,(x_1, y_1)\,\Delta x = y_1 + (x_1 + y_1)\,\Delta x = 1.2 + (1.1 + 1.2)\,(0.1) = 1.43 \\ x_3 = x_2 + \Delta x & y_3 = y_2 + m\,(x_2, y_2)\,\Delta x = y_2 + (x_2 + y_2)\,\Delta x = 1.43 + (1.2 + 1.43)\,(0.1) = 1.693 \end{array}$$

So $y(1.3) \approx 1.693$.

□

3. Find an explicit solution of the following (separable) differential equation.

$$2x - e^{2y} y' = 0$$

- We have $M(x) = 2x$ and $N(y) = -e^{2y}$, as an implicit solution we'll have

$$\int 2x dx - \int e^{2y} dy = C \qquad \Rightarrow \quad x^2 - \frac{1}{2} e^{2y} = C$$

Solving for y we obtain

$$y = \frac{1}{2} \ln \left| 2x^2 - 2C \right|$$

□

4. Solve the following initial value problem

$$y' - \frac{3}{x} y = x \qquad , \qquad y(1) = 2$$

- This is a first order linear equation with $p(x) = -3/x$ and $g(x) = x$. So the general solution is

$$\begin{aligned} \mu(x) &= \exp\left(\int p(x)dx\right) = \exp\left(\int -\frac{3}{x} dx\right) = \exp\left(-3 \ln |x|\right) = x^{-3} \\ y(x) &= \frac{1}{\mu} \int \mu g dx + \frac{C}{\mu} = \frac{1}{x^{-3}} \int x^{-3}(x) \, dx + \frac{C}{x^{-3}} = x^3 \int x^{-2} dx + Cx^{-3} \\ &= x^3 \left(\frac{1}{-1} x^{-1} \right) + Cx^3 = -x^2 + Cx^3 \end{aligned}$$

Plugging the general solution into the initial condition yields

$$\begin{aligned} 2 &= y(1) = \left[-x^2 + Cx^3 \right] \Big|_{x=1} = -1 + C \qquad \Rightarrow \quad C = 3 \\ \Rightarrow & \quad y = -x^2 + 3x^3 \end{aligned}$$

□

5. Show that the following equation is exact

$$\frac{y}{x} + 2x + \ln |x| \frac{dy}{dx} = 0$$

and then find the explicit solution of this differential equation.

- For this problem, we have $M(x, y) = y/x + 2x$ and $N(x, y) = \ln |y|$. We have

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}$$

so the equation is exact. Let's now find an explicit solution to the following initial value problem

$$\frac{y}{x} + 2x + \ln |x| \frac{dy}{dx} = 0$$

$$\begin{aligned} \Phi(x, y) &= \int M \partial x + C_1(y) = \int \left(\frac{y}{x} + 2x \right) \partial x + C_1(y) = y \ln |x| + x^2 + C_1(y) \\ &= \int N \partial y + C_2(x) = \int \ln |x| \, \partial y + C_2(x) = \ln |x| \, y + C_2(x) \end{aligned}$$

The consistency for these two expression for Φ requires $C_1(y) = 0$ and $C_2(x) = x^2$. Thus, $\Phi = y \ln |x| + x^2$. Our implicit solution is thus

$$y \ln |x| + x^2 = C \qquad \Rightarrow \quad y = \frac{C - x^2}{\ln |x|}$$

□

6. Due to it radioactivity Carbon 14 decays according to a simple first order linear ODE.

$$\frac{dQ}{dt} = -kQ$$

(here $Q(t)$ represents the quantity of Carbon 14 at time t). If the half-life of Carbon 14 (the time it takes to decay to 1/2 quantity) is 5730 years, find $Q(t)$ for a specimen that originally contained 10 grams of Carbon 14.

- The general solution to the differential equation

  $$\frac{dQ}{dt} + kQ = 0$$

  is readily seen to be (since it is first order linear)

  $$Q(t) = \frac{1}{\exp\left(\int k\ dt\right)} \int 0 \cdot \exp\left(\int k\ dt\right) + \frac{C}{\exp\left(\int k\ dt\right)} = 0 + Ce^{-kt} = Ce^{-kt}$$

  The constant of integration C corresponds to the initital quantity at time $t = 0$

  $$10 = Q(0) = Ce^{-k \cdot 0} \quad \Rightarrow \quad C = 10$$

  To figure out k, we use the half-life information. In 5730 years $Q(0)$ should be reduced to $\frac{1}{2}Q(0)$. Thus,

  $$\frac{1}{2}(10) = 10e^{-5730k} \quad \Rightarrow \quad e^{-5730k} = \frac{1}{2}$$

  or

  $$k = -\frac{1}{5730} \ln\left(\frac{1}{2}\right) = \frac{\ln|2|}{5730}$$

  Thus,

  $$Q(t) = 10 \exp\left(\frac{\ln|2|}{5730} t\right)$$