# Second Order Linear Equations, General Theory

## 1. Standard Form

A second order linear differential equation is a differential equation of the form

$$(11.1) \qquad A(x)y'' + B(x)y' + C(x)y = D(x) \quad .$$

(Here $A, B, C$ and $D$ are certain prescribed functions of $x$.)

As in the case of first order linear equations, in any interval where $A(x) \neq 0$, we can replace such an equation by an equivalent one in **standard form**:

$$(11.2) \qquad y'' + p(x)y' + q(x)y = g(x)$$

where

$$(11.3) \qquad \begin{aligned} p(x) &= \frac{B(x)}{A(x)} \\ q(x) &= \frac{C(x)}{A(x)} \\ g(x) &= \frac{D(x)}{A(x)} \end{aligned}$$

## 2. Homogeneous vs. Non-homogeneous Linear Differential Equations

In the development that follows it will be important to distinguish between the case when the right hand side of

$$(11.4) \qquad y'' + p(x)y' + q(x)y = g(x)$$

is zero or non-zero. We shall say that a second order linear ODE is **homogeneous** if it can be written in the form

$$(11.5) \qquad y'' + p(x)y' + q(x)y = 0$$

otherwise (if $g(x) \neq 0$) we shall say that it is **non-homogeneous**. Note that this terminology is completely unrelated to homogeneous equations of degree zero (the topic of the preceding lecture).

## 3. Differential Operator Notation

Consider the general second order linear differential equation

$$(11.6) \qquad \phi'' + p(x)\phi' + q(x)\phi = g(x) \quad .$$

We shall often write differential equations like this as

$$(11.7) \qquad L[\phi] = g(x)$$

where $L$ is the **linear differential operator**

$$(11.8) \qquad L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) \quad .$$

That is to say, $L$ is the operator that acts on a function $\phi$ by

(11.9)
$$\begin{aligned} L[\phi] &= \left( \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) \right) \phi \\ &= \frac{d^2\phi}{dx^2} + p(x)\frac{d\phi}{dx} + q(x)\phi \quad . \end{aligned}$$

## 4. General Theorems

The following theorem tells us the conditions for the existence and uniqueness of solutions of a second order linear differential equation.

THEOREM 11.1. *If the functions $p$, $q$ and $g$ are continuous on an open interval $I \subset \mathbb{R}$ containing the point $x_o$, then in some interval about $x_o$ there exists a unique solution $y = \phi(x)$ to the differential equation*

(11.10)
$$y'' + p(x)y' + q(x)y = g(x)$$

*satisfying the prescribed initial conditions*

(11.11)
$$\begin{aligned} y(x_o) &= y_o \\ y'(x_o) &= y'_o \quad . \end{aligned}$$

Note how this theorem is analogous to the corresponding theorem for first order linear ODE's. Note also that the conditions for existence and uniqueness are fairly lax - all we require is the continuity of the functions $p$, $q$, and $g$ around a given initial point. Finally, we note that the form of the initial conditions involves the specification of **both** $y(x)$ and its derivative $y'(x)$ at an initial point $x_o$.

I should also point out that the preceding theorem does not address the issue of how to construct a solution of a second order linear ODE. Indeed, the actual construction of solutions to second order linear ODE is sufficiently complicated to that we shall spend 90% of the remaining lectures on techniques of solution. The next two theorems at least tell us the basic ingredients for a general solution of a second order linear ODE.

THEOREM 11.2. *(The Superposition Principle) If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the differential equation*

(11.12)
$$L[y] = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

*then any linear combination*

(11.13)
$$y = c_1 y_1(x) + c_2 y_2(x)$$

*of $y_1(x)$ and $y_2(x)$, where $c_1$ and $c_2$ are constants, is also a solution of (11.12).*

*Proof.*

(11.14)
$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= \frac{d^2}{dx^2}(c_1 y_1 + c_2 y_2) + p(x)\frac{d}{dx}(c_1 y_1 + c_2 y_2) + q(x)(c_1 y_1 + c_2 y_2) \\ &= c_1\left(\frac{d^2 y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1\right) + c_2\left(\frac{d^2 y_3}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2\right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 \\ &= 0 \end{aligned}$$

$\square$

The fact that a linear combination of solutions of a **linear, homogeneous differential equation** is also a solution is extremely important. The theory of linear homogeneous equations, including differential equations involving higher derivatives depends strongly on the superposition principle.

EXAMPLE 11.3.

(11.15) $$y_1(x) = \cos(x)$$

and

(11.16) $$y_2(x) = \sin(x)$$

are both solutions of

(11.17) $$y'' + y = 0.$$

It is easy to check that any linear combination of $y_1$ and $y_2$ is also a solution.

EXAMPLE 11.4.

(11.18) $$y_1(x) = 1$$

and

(11.19) $$y_2(x) = x^{1/2}$$

are both solutions of

(11.20) $$yy'' + (y')^2 = 0.$$

However, it is easy to check that $y_1 + y_2 = 1 + \sqrt{x}$ is not a solution of (11.20). The reason for this lies in the fact that (11.20) is not linear.

Given two solutions $y_1$ and $y_2$ of a second order linear homogeneous differential equation

(11.21) $$L[y] = 0 \quad,$$

we can construct an infinite number of other solutions

(11.22) $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

by letting $c_1$ and $c_2$ run through $\mathbb{R}$. The following question then arises: are **all** the solutions of (11.21) capable of being expressed in form (11.22) for some choice of $c_1$ and $c_2$?

This will not always be the case; and so we shall say that two solutions $y_1$ and $y_2$ form a **fundamental set** of solutions to (11.21) if every solution of (11.21) can be expressed as a linear combination of $y_1$ and $y_2$.

THEOREM 11.5. *If $p$ and $q$ are continuous on an open interval $I = (\alpha, \beta)$ and if $y_1$ and $y_2$ are solutions of the differential equation*

(11.23) $$L[y] = y'' + p(x)y' + q(x)y = 0$$

*satisfying*

(11.24) $$W(y_1, y_2) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0$$

*at every point $x \in I$, then any other solution of (11.23) on the interval $I$ can be expressed uniquely as a linear combination of $y_1$ and $y_2$.*

*Proof.*

Let $y_1$ and $y_2$ be two given solutions on an interval $I$ and let $Y$ be an any other solution on $I$. Choose a point $x_o \in I$. From our basic uniqueness and existence theorem (Theorem 3.2), we know that there is only solution $y(x)$ of (11.23) such that

(11.25) $$\begin{aligned} y(x_o) &= Y(x_o) \\ y'(x_o) &= Y'(x_o) \end{aligned} \quad.$$

namely, $Y(x)$. Therefore if we can show that a solution of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies the initial conditions (11.25), then we must have $Y(x) = c_1 y_1(x) + c_2 y_2(x)$ and so $Y(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$.

Thus, we now seek to define constants $c_1$ and $c_2$ so that these initial conditions can be matched. We thus set

$$(11.26) \qquad \begin{aligned} c_1 y_1(x_o) + c_2 y_2(x_o) &= y_o \\ c_1 y_1'(x_o) + c_2 y_2'(x_o) &= y_o' \quad . \end{aligned}$$

This is just a series of two equations with two unknowns. Solving the first equation for $c_1$ yields

$$(11.27) \qquad c_1 = \frac{y_o - c_2 y_2(x_o)}{y_1(x_o)} \quad .$$

Plugging this into the second equation yields

$$(11.28) \qquad \frac{y_o - c_2 y_2(x_o)}{y_1(x_o)} y_1'(x_o) + c_2 y_2'(x_o) = y_o'$$

or

$$(11.29) \qquad y_o y_1'(x_o) - c_2 y_2(x_o) y_1'(x_o) + c_2 y_1(x_o) y_2'(x_o) = y_1(x_o) y_o'$$

or

$$(11.30) \qquad c_2 = \frac{y_1(x_o) y_o' - y_1'(x_o) y_o}{y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o)} \quad .$$

Plugging this expression for $c_2$ into (11.27) yields

$$(11.31) \qquad c_1 = \frac{y_o y_2'(x_o) - y_2(x_o) y_o'}{y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o)} \quad .$$

Thus, we can solve for $c_1$ and $c_2$ whenever the denominator

$$(11.32) \qquad W(y_1, y_2) = y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o)$$

does not vanish. Thus, so long as $y_1$ and $y_2$ satisfy (11.23) we can always express any solution as a linear combination of $y_1$ and $y_2$.                                                                 □

Remark: The quantity

$$(11.33) \qquad W(y_1, y_2) = y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o)$$

is called the **Wronskian** of $y_1$ and $y_2$.

EXAMPLE 11.6. Show that

$$(11.34) \qquad y_1(x) = \cos(x)$$

and

$$(11.35) \qquad y_2(x) = \sin(x)$$

are form a set of fundamental solutions to the differential equation

$$(11.36) \qquad y'' + y = 0 \quad .$$

We simply have to check that the Wronskian does not vanish:

$$(11.37) \qquad \begin{aligned} W(y_1, y_2) &= y_1(x_o) y_2'(x_o) - y_1'(x_o) y_2(x_o) \\ &= \cos(x)\,(\cos(x)) - (-\sin(x))\sin(x) \\ &= 1 \\ &\neq 0 \quad . \end{aligned}$$

Since the Wronskian does not vanish, $y_1$ and $y_2$ must be linearly independent.