LECTURE 12

Reduction of Order

Recall that the general solution of a second order homogeneous linear differential equation

$$L[y] = y'' + p(x)y' + q(x)y = 0 \tag{1}$$

is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{2}$$

where y_1 and y_2 are any two solutions such that

$$W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0 \quad . \tag{3}$$

In this section we shall assume that we have already found one solution y_1 of (1) and that we are seeking to find another solution y_2 so that we can write down the general solution as in (2).

So suppose we have one non-trivial solution $y_1(x)$ of (1) and suppose there is another solution of the form

$$y_2(x) = v(x)y_1(x) \quad .$$

Then

$$\begin{aligned} W[y_1, y_2] &= y_1 y_2' - y_1' y_2 \\ &= y_1(v'y_1 + vy_1') - y_1'(vy_1) \\ &= (y_1)^2 v' \\ &\neq 0 \end{aligned}$$

unless $v' = 0$. Thus, any solution we construct by multiplying our given solution $y_1(x)$ by a non-constant function $v(x)$ will give us another linearly independent solution.

The question we now wish to address is: how does one find an appropriate function $v(x)$?

Certainly, we want to choose $v(x)$ so that $y_2(x) = v(x)y_1(x)$ satisfies (1). So let us insert $y(x) = v(x)y_1(x)$ into (1):

$$\begin{aligned} 0 &= \tfrac{d^2}{dx^2}(vy_1) + p(x)\tfrac{d}{dx}(vy_1) + q(vy_1) \\ &= v''y_1 + 2v'y_1' + vy_1'' + p(x)v'y_1 + p(x)vy_1' + qvy_1 \\ &= v(y_1'' + p(x)y_1' + q(x)y_1) + v''y_1 + (2y_1' + p(x)y_1)v' \end{aligned} \tag{4}$$

The first term vanishes since y_1 is a solution of (1), so $v(x)$ must satisfy

$$0 = y_1 v'' + (2y_1' + p(x)y_1)v'$$

or

$$v'' + \left(p(x) + \frac{2y_1'}{y_1}\right)v' = 0 \quad . \tag{5}$$

Now set

$$u(x) = v'(x) \quad . \tag{6}$$

Then we have

$$u' + \left(p(x) + \frac{2y_1'(x)}{y_1(x)}\right)u = 0 \quad . \tag{7}$$

This is a first order linear differential equation which we know how to solve. Its general solution is

$$\begin{aligned} u(x) &= C \exp\left[-\int^x \left(p(t) + \frac{2y_1'(t)}{y_1(t)}\right) dt\right] \\ &= C \exp\left[-\int^x (p(t))\, dt - 2\int^x \frac{y_1'(t)}{y_1(t)} dt\right] \end{aligned} \tag{8}$$

Now note that

$$\frac{d}{dt} \ln [y_1(t)] = \frac{y_1'(t)}{y_1(t)} \quad ,$$

so

$$\begin{aligned} \exp\left[-2\int^x \frac{y_1'(t)}{y_1(t)} dy\right] &= \exp\left[-2\int^x \frac{d}{dt}\left(\ln [y_1(t)]\right) dt\right] \\ &= \exp\left[-2\ln [y_1(x)]\right] \\ &= \exp\left[\ln\left[(y_1(x))^{-2}\right]\right] \\ &= \frac{1}{(y_1(x))^2} \quad . \end{aligned}$$

Thus, (8) can be written as

$$u(x) = \frac{C}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right] \quad .$$

Now recall from (6) that $u(x)$ is the derivative of the factor v(x) which we originally sought out to find. So

$$\begin{aligned} v(x) &= \int^x u(t)\, dt + D \\ &= \int^x \left[\frac{C}{(y_1(t))^2} \exp\left[-\int^t p(t')dt'\right]\right] + D \end{aligned}$$

It is not too difficult to convince oneself that it is not really necessary to carry along the constants of integration C and D. For the constant D can be absorbed into the constant c_1 of the general solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$, while the factor C can be absorbed into the constant c_2. Thus, without loss of generality, we can take $C = 1$ and $D = 0$. So given one solution $y_1(x)$ of (1), a second solution $y_2(x)$ of (1) can be formed by computing

$$v(x) = \int^x u(t)\, dt$$

where

$$u(x) = \frac{1}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right]$$

and then setting

$$y_2(x) = v(x) y_1(x) \quad .$$

The general solution of (1) is then

$$y(x) = c_1 y_1(x) + c_2 v(x) y_1(x) \quad .$$

This technique for constructing the general solution from single solution of a second order linear homogeneous differential equation is called **reduction of order**.

For those of you who like nice tidy formulae we can write

$$\boxed{y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[-\int^s p(t)dt\right] ds}$$

for the second solution.

EXAMPLE 12.1.

$$y_1(x) = e^{-x}$$

is one solution of

$$y'' + 2y' + y = 0 \quad .$$

Find another linearly independent solution.

Well, $p(x) = 2$, so

$$\begin{aligned} u(x) &= \tfrac{C}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right] \\ &= \tfrac{C}{e^{-2x}} \exp\left[-2x\right] \\ &= Ce^{2x}e^{-2x} \\ &= C \end{aligned}$$

So

$$v(x) = \int^x u(t)\,dt = \int^x C dt = Cx.$$

Thus,

$$y_2(x) = v(x)y_1(x) = Cxe^{-x} \quad .$$

Example 12.2.

$$y_1(x) = x$$

is a solution of

$$x^2y'' + 2xy' - 2y = 0 \quad .$$

Use reduction of order to find the general solution.

Well, we first put the differential equation in standard form:

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0 \quad .$$

Thus,

$$p(x) = \frac{2}{x} \quad , \quad q(x) = \frac{-2}{x^2} \quad .$$

We first compute $u(x)$.

$$\begin{aligned} u(x) &= \tfrac{C}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right] \\ &= \tfrac{C}{x^2} \exp\left[\int^x -\tfrac{2}{t}dt\right] \\ &= \tfrac{C}{x^2} \exp\left[-2\ln[x]\right] \\ &= \tfrac{C}{x^2} x^{-2} \\ &= Cx^{-4} \quad . \end{aligned}$$

So

$$\begin{aligned} v(x) &= \int^x u(t)dt \\ &= \int^x Ct^{-4}dt \\ &= -\tfrac{C}{3}x^{-3} \end{aligned}$$

and

$$y_2(x) = v(x)y_1(x) = -\frac{C}{3x^3}x = C'x^{-2} \quad .$$

The general solution of the original differential equation is thus

$$y(x) = c_1y_1(x) + c_2y_2(x) = c_1x + c_2x^{-2} \quad .$$