LECTURE 14

Homogeneous Equations with Constant Coefficients, Cont'd

Recall that the general solution of a 2^{nd} order linear homogeneous differential equation

$$L[y] = y'' + p(x)y' + q(x)y = 0 \tag{14.1}$$

is always a linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{14.2}$$

of two linearly independent solutions y_1 and y_2, and we've seen that if we're given one solution $y_1(x)$ we can compute a second linearly independent solution using the method of reduction of order. We will now turn to the problem of actually finding a single solution $y_1(x)$ of (14.1).

We let us now return to the special case of a homogeneous second order linear differential equation with constant coefficients; i.e., differential equations of the form

$$y'' + py' + qy = 0 \tag{14.3}$$

where p and q are constant.

We saw in Lecture 11, that one can construct solutions of the differential equation (14.3) by looking for solutions of the form

$$y(x) = e^{\lambda x} \quad . \tag{14.4}$$

Let us recall that construction. Plugging (14.4) into (14.3) yields

$$0 = \lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + q e^{\lambda x} = \left(\lambda^2 + p\lambda + q\right) e^{\lambda x} \quad . \tag{14.5}$$

Since the exponential function $e^{\lambda x}$ never vanishes we must have

$$\lambda^2 + p\lambda + q = 0 \quad . \tag{14.6}$$

Equation (14.6) is called the **characteristic equation** for (14.3) since for any λ satisfying (14.6) we will have a solution $y(x) = e^{\lambda x}$ of (14.3).

Now because (14.6) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

$$\lambda^2 + p\lambda + q = 0 \qquad \Rightarrow \qquad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} \quad . \tag{14.7}$$

Note that a root λ of (14.6) need not be a real number. Indeed, if $p^2 - 4q < 0$, then in order to compute λ via the Quadratic Formula we have to take the square root of a negative number and that forces us into the realm of complex numbers. We shall postpone temporarily the case when a root λ is complex and first discuss the case when the roots of (14.6) are all real. This requires $p^2 - 4q \geq 0$.

Case (i): $p^2 - 4q > 0$

Because $p^2 - 4q$ is positive, $\sqrt{p^2 - 4q}$ is a positive real number and

$$\begin{aligned} \lambda_+ &= \tfrac{-p + \sqrt{p^2 - 4q}}{2} \\ \lambda_- &= \tfrac{-p - \sqrt{p^2 - 4q}}{2} \end{aligned} \tag{14.8}$$

are distinct real roots of (14.6). Thus,

$$\begin{aligned} y_1 &= e^{\lambda_+ x} \\ y_2 &= e^{\lambda_- x} \end{aligned} \tag{14.9}$$

will both be solutions of (14.3). Noting that

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_1' y_2 \\ &= \lambda_- e^{\lambda_+ x} e^{\lambda_- x} - \lambda_+ e^{\lambda_+ x} e^{\lambda_- x} \\ &= (\lambda_- - \lambda_+)\, e^{(\lambda_+ + \lambda_-) x} \\ &= \frac{\sqrt{p^2 - 4q}}{a} e^{-\frac{b}{a} x} \end{aligned} \tag{14.10}$$

is non-zero, we conclude that if $p^2 - 4q \neq 0$, then the roots (14.8) furnish two linearly independent solutions of (14.3) and so the general solution is given by

$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x} \quad . \tag{14.11}$$

Case (ii): $p^2 - 4q = 0$

If $p^2 - 4q = 0$, however, this construction only gives us one distinct solution; because in this case $\lambda_+ = \lambda_-$. To find a second fundamental solution we must use the method of Reduction of Order.

So suppose $y_1(x) = e^{-\frac{p}{2} x}$ is the solution corresponding to the root

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2} = \frac{-p \pm 0}{2} = \frac{-p}{2}$$

of

$$\lambda^2 + p\lambda - q = 0 \quad , \quad p^2 - 4q = 0.$$

Then the Reduction of Order formula gives us a second linearly independent solution

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[\int^s -p(t) dt\right] ds$$

gives us a second linearly independent solution. Plugging in $y_1(x) = e^{-\frac{p}{2} x}$ and $p(t) = p$, yields

$$\begin{aligned} y_2(x) &= e^{-\frac{p}{2} x} \int^x \frac{1}{\left(e^{-\frac{p}{2} s}\right)^2} \exp\left[\int^s -p dt\right] ds \\ &= e^{-\frac{p}{2} x} \int^x \frac{1}{e^{-ps}} \exp\left[-ps\right] ds \\ &= e^{-\frac{p}{2} x} \int^x e^{ps} e^{-ps} ds \\ &= e^{-\frac{p}{2} x} \int^x ds \\ &= x e^{-\frac{p}{2} x} \\ &= x y_1(x) \end{aligned}$$

In summary, for the case when $p^2 - 4q = 0$, we only have one root of the characterisitic equation, and so we get only one distinct solution $y_1(x)$ of the original differential equation by solving the characteristic equation for λ. To get a second linearly solution we must use the Reduction of Order formula; however, the result will always be the same: **the second linearly independent solution will always be x times the solution $y_1(x) = e^{-\frac{p}{2} x}$**. Thus, the general solution in this case will be

$$y(x) = c_1 e^{-\frac{p}{2} x} + c_2 x e^{-\frac{p}{2} x} \quad , \quad \text{if } p^2 - 4q = 0.$$

We now turn to the third and last possibility.

Case (iii): $p^2 - 4q < 0$

In this case

$$\sqrt{p^2 - 4q} \tag{14.12}$$

will be undefined unless we introduce complex numbers. But when we set

$$\sqrt{-1} = i \tag{14.13}$$

we have

$$\sqrt{p^2 - 4q} = \sqrt{(-1)(4q - p^2)} = \sqrt{-1}\sqrt{4q - p^2} = i\sqrt{4q - p^2} \quad . \tag{14.14}$$

The square root on the right hand side is well-defined since $4q - p^2$ is a positive number. Thus,

$$\lambda_{\pm} = \frac{-p \pm i\sqrt{4q - p^2}}{2} = \alpha \pm i\beta \tag{14.15}$$

where

$$\alpha = -\frac{b}{2} \quad , \quad \beta = \frac{\sqrt{4q - p^2}}{2} \quad , \tag{14.16}$$

will be a complex solution of (14.6) and

$$y(x) = c_1 e^{\alpha x + i\beta x} + c_2 e^{\alpha x - i\beta x} \tag{14.17}$$

would be a solution of (14.3) if we could make sense out the notion of an exponential function with a complex argument.

Thus, we must address the problem of ascribing some meaning to

$$e^{\alpha x + i\beta x} \tag{14.18}$$

as a function of x. To ascribe some sense to this expression we considered the Taylor series expansion of e^x

$$\begin{aligned} e^x &= 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots \\ &= \textstyle\sum_{i=0}^{\infty} \frac{1}{i!} x^i \end{aligned} \tag{14.19}$$

Now although we do not yet understand what $e^{\alpha x + i\beta x}$ means, we can nevertheless substitute $\alpha x + i\beta$ for x on the right hand side of (14.19), and get a well defined series with values in the complex numbers. One can show that this series converges for all α, β and x. We thus take

$$e^{\alpha x + i\beta} = \lim_{n \to \infty} \sum_{i=0}^{n} \frac{1}{i!} (\alpha x + i\beta x)^i \tag{14.20}$$

which agrees with (14.19) when $\beta = 0$.

One can also show that

$$e^{\alpha x + i\beta x} = e^{\alpha x} e^{i\beta x}. \tag{14.21}$$

Thus, when $p^2 - 4q = 0$, we have two complex valued solutions to (14.3)

$$y_1(x) = e^{\alpha x} e^{i\beta x} \quad \text{and} \quad y_2(x) = e^{\alpha x} e^{-i\beta x} \quad , \tag{14.22}$$

where

$$\alpha = \frac{-p}{2} \quad , \quad \beta = \frac{\sqrt{4q - p^2}}{2} \quad . \tag{14.23}$$

A general solution of (14.3) would then be

$$y(x) = c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x} e^{i\beta x}. \tag{14.24}$$

However, this is rarely the form in which one wants a solution of (14.3). One would prefer solutions that are **real-valued functions of** x rather that complex-valued functions of x. But these can be had as well, since if $z = x + iy$ is a complex number, then

$$\begin{aligned} Re(z) &= \tfrac{1}{2}(z+\bar{z}) = x \\ Im(z) &= \tfrac{1}{2i}(z-\bar{z}) = y \end{aligned} \tag{14.25}$$

are both real numbers. Applying the Superposition Principle, it is easy to see that if

$$y(x) = e^{\alpha x}e^{i\beta x} \tag{14.26}$$

and

$$\bar{y}(x) = e^{\alpha x}e^{-i\beta x} \tag{14.27}$$

are two complex-valued solutions of (14.3), then

$$y_r(x) = \frac{1}{2}(y(x)+\bar{y}(x)) = e^{\alpha x}\left(\frac{e^{i\beta x}+e^{-i\beta x}}{2}\right) \tag{14.28}$$

and

$$y_i(x) = \frac{1}{2i}(y(x)-\bar{y}(x)) = e^{\alpha x}\left(\frac{e^{i\beta x}-e^{-i\beta x}}{2i}\right) \tag{14.29}$$

are both real-valued solutions of (14.3).

Let us now compute the series expansion of

$$\frac{e^{ix}+e^{-ix}}{2} \tag{14.30}$$

and

$$\frac{e^{ix}-e^{-ix}}{2i} \quad . \tag{14.31}$$

(14.32)

$$\begin{aligned} \tfrac{1}{2}\left(e^{ix}+e^{-ix}\right) &= \tfrac{1}{2}\left(1+(ix)+\tfrac{1}{2!}(ix)^2+\tfrac{1}{3!}(ix)^3+\cdots\right) \\ &\quad +\tfrac{1}{2}\left(1+(-ix)+\tfrac{1}{2!}(-ix)^2+\tfrac{1}{3!}(-ix)^3+\cdots\right) \\ &= \left(1-\tfrac{1}{2!}x^2+\tfrac{1}{4!}x^4+\cdots\right) \end{aligned}$$

The expression on the right hand side is readily identified as the Taylor series expansion of $\cos(x)$. We thus conclude

$$\boxed{\cos(x) = \frac{e^{ix}+e^{-ix}}{2} \quad .} \tag{14.33}$$

Similarly, one can show that

$$\boxed{\sin(x) = \frac{e^{ix}-e^{-ix}}{2i} \quad .} \tag{14.34}$$

On the other hand, if one adds (14.33) to i times (14.34) one gets

$$\cos(x)+i\sin(x) = \frac{e^{ix}+e^{-ix}}{2}+i\frac{e^{ix}-e^{-ix}}{2i} = \frac{e^{ix}+e^{-ix}+e^{ix}-e^{-ix}}{2} = e^{ix} \tag{14.35}$$

or

$$\boxed{e^{ix} = \cos(x)+i\sin(x)} \tag{14.36}$$

Thus, the real part of e^{ix} is $\cos(x)$, while the pure imaginary part of e^{ix} is $\sin(x)$.

We now have a means of interpreting the function

$$e^{\alpha x+i\beta x} \tag{14.37}$$

in terms of elementary functions (rather than as a power series); namely,

$$e^{\alpha x+i\beta x}=e^{\alpha x}e^{i\beta x}=e^{\alpha x}\left(\cos(\beta x)+i\sin(\beta x)\right). \tag{14.38}$$

Thus,

$$\boxed{\begin{array}{rcl} Re\left[e^{\alpha x+i\beta x}\right] & = & e^{\alpha x}\cos(\beta x) \quad , \\ Im\left[e^{\alpha x+i\beta x}\right] & = & e^{\alpha x}\sin(\beta x) \quad . \end{array}} \tag{14.39}$$

I now want to show how (14.33) and (14.34) allow us to write down the general solution of a differential equation of the form

$$y''+py'+qy=0 \quad , \quad p^2-4q<0 \tag{14.40}$$

as a linear combination of real-valued functions.

Now when $p^2-4q<0$, then

$$\lambda_{\pm}=\frac{-p\pm i\sqrt{4q-p^2}}{2}=\alpha\pm i\beta \tag{14.41}$$

are the (complex) roots of the characteristic equation

$$\lambda^2+p\lambda+q=0 \tag{14.42}$$

corresponding to (14.40) and

$$y_{\pm}(x)=e^{\alpha x\pm i\beta} \tag{14.43}$$

are two (complex-valued) solutions of (14.40). But since (14.40) is linear, since y_+ and y_- are solutions so are

$$\begin{array}{rcl} y_1(x) & = & \frac{1}{2}\left(y_+(x)+y_-(x)\right) \\ & = & \frac{1}{2}\left(e^{\alpha x+i\beta x}+e^{\alpha x-i\beta x}\right) \\ & = & e^{\alpha x}\left(\frac{e^{i\beta x}+e^{-i\beta x}}{2}\right) \\ & = & e^{\alpha x}\cos(\beta x) \end{array} \tag{14.44}$$

and

$$\begin{array}{rcl} y_2(x) & = & \frac{1}{2i}\left(y_+(x)-y_-(x)\right) \\ & = & \frac{1}{2i}\left(e^{\alpha x+i\beta x}-e^{\alpha x-i\beta x}\right) \\ & = & e^{\alpha x}\left(\frac{e^{i\beta x}-e^{-i\beta x}}{2i}\right) \\ & = & e^{\alpha x}\sin(\beta x) \quad . \end{array} \tag{14.45}$$

Note that y_1 and y_2 are both **real-valued functions**.

We conclude that if the characteristic equation corresponding to

$$y''+py'+qy=0 \tag{14.46}$$

has two complex roots

$$\lambda=\alpha\pm i\beta \tag{14.47}$$

then the general solution is

$$\boxed{y(x)=c_1e^{\alpha x}\cos(\beta x)+c_2e^{\alpha x}\sin(\beta x) \quad .} \tag{14.48}$$

EXAMPLE 14.1. The differential equation

$$y'' - 2y' - 3y \tag{14.49}$$

has as its characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0 \quad . \tag{14.50}$$

The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{2 \pm \sqrt{4+12}}{2} \\ &= 3, -1 \quad . \end{aligned} \tag{14.51}$$

These are distinct real roots, so the general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-x} \quad . \tag{14.52}$$

EXAMPLE 14.2. The differential equation

$$y'' + 4y' + 4y = 0 \tag{14.53}$$

has

$$\lambda^2 + 4\lambda + 4 = 0 \tag{14.54}$$

as its characteristic equation. The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{-4 \pm \sqrt{16-16}}{2} \\ &= -2 \quad . \end{aligned} \tag{14.55}$$

Thus we have a double root and the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} \quad . \tag{14.56}$$

EXAMPLE 14.3. The differential equation

$$y'' + y' + y = 0 \tag{14.57}$$

has

$$\lambda^2 + \lambda + 1 = 0 \tag{14.58}$$

as its characteristic equation. The roots of the characteristic equation are

$$\begin{aligned} \lambda &= \tfrac{-1 \pm \sqrt{1-4}}{2} \\ &= -\tfrac{1}{2} \pm i \tfrac{\sqrt{3}}{2} \end{aligned} \tag{14.59}$$

and so the general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2} x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2} x\right) \quad . \tag{14.60}$$