LECTURE 16

The Nonhomogeneous Problem

We now consider differential equations of the form

$$y'' + p(x)y' + q(x)y = g(x) \tag{16.1}$$

where $g(x) \neq 0$.

Obvious things that we'd like to know are

- how to construct solutions; and
- how to know if we have all the solutions.

To this end, it certainly would be nice to have something like the Superposition Principle at our disposal. However, for non-homogeneous linear differential equations the Superposition Principle can not be applied. To see this, suppose $Y_1(x)$ and $Y_2(x)$ are solutions of (16.1). If the Superposition Principle were valid then $Y(x) = c_1Y_1(x) + c_2Y_2(x)$ would also be a solution. But for this $Y(x)$

$$\begin{aligned} Y'' + p(x)Y' + q(x)Y &= c_1Y_1'' + c_2Y_2'' + p(x)\left(c_1Y_1' + c_2Y_2'\right) + q(x)\left(c_1Y_1 + c_2Y_2\right) \\ &= c_1\left(Y_1'' + p(x)Y_1' + q\left(x\right)Y_1\right) + c_2\left(Y_2'' + p(x)Y_2' + q(x)Y_2\right) \\ &= c_1g(x) + c_2g(x) \\ &= \left(c_1 + c_2\right)g(x) \\ &\neq g(x) \end{aligned}$$

Thus, if $y_1(x)$ and $y_2(x)$ satisfy (16.1) then a linear combinattion of y_1 and y_2 doesn's satisfy the same equation. So we can't make more solutions out of two independent solutions.

However, the calculation carried above, nevertheless, leads us to a way of constructing the general solution to (16.1). Let $Y_1(x)$ and $Y_2(x)$ be any two solutions of (16.1) and consider the function $\Delta Y(x)$ defined as the difference of $Y_1(x)$ and $Y_2(x)$:

$$\Delta Y(x) = Y_1(x) - Y_2(x)\,.$$

Applying the calculation above with $c_1 = 1$ and $c_2 = -1$ we see that $Y(x)$ obeys

$$Y'' + p(x)Y' + q(x)Y = (1-1)g(x) = 0$$

Thus, the difference of any two solutions of (16.1) must be a solution of the corresponding homogeneous equation.

Now let's assume that we've solved the correpsponding homogeneous equation

$$(\Delta y)'' + p(x)\left(\Delta Y\right)' + q(x)\Delta Y = 0. \tag{16.2}$$

From the general theory of homogeneous second order linear equations, we know that every solution can be expressed as a linear combination of two linearly independent solutions. Suppose then that $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of (16.2) so that any solution of (16.2) can be expressed in the form $c_1y_1(x) + c_2y_2(x)$. This implies in particular that the function $\Delta Y(x) = Y_1(x) - Y_2(x)$ must be expressible in the form $c_1y_1(x) + c_2y_2(x)$ (since as we have seen $y(x)$ satisfies (16.2)). Thus, we have

$$Y_1(x) - Y_2(x) = y(x) = c_1y_1(x) + c_2y_2(x) \tag{16.3}$$

(Note that the functions $Y_1(x)$ and $Y_2(x)$ on the left hand side are solutions of the non-homogeneous equation (16.1) and the functions $y_1(x)$ and $y_2(x)$ on the right hand side are two linearly independent solutions of the corresponding homogeneous equation (16.2)). Now, in this calculation both $Y_1(x)$ and $Y_2(x)$ are arbitrary solutions of the non-homogeneous equation (16.1). Let's now interprete $Y_2(x)$ as some fixed solution $Y_p(x)$ of (16.1) and interprete $Y_1(x)$ as representing any other solution $Y(x)$ of (16.1). Then equation (16.3) becomes

$$Y(x) - Y_p(x) = c_1y_1(x) + c_2y_2(x)$$

or

$$Y(x) = Y_p(x) + c_1y_1(x) + c_2y_2(x).$$

Thus, every solution of the non-homogeneous equation (16.1) can be expressed in terms of a single solution $Y_p(x)$ and a linear combination of two linearly independent solutions of the corresponding homogeneous problem (16.2).

The following two theorems summarize these results and provide the foundation by which we construct solutions of such non-homogeneous second order ODEs.

THEOREM 16.1. *If Y_1 and Y_2 are two solutions of the nonhomogeneous equation*

$$y'' + p(x)y' + q(x)y = g(x) \quad , \tag{16.4}$$

then their difference

$$Y(x) = Y_1(x) - Y_2(x) \tag{16.5}$$

is a solution of the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \quad . \tag{16.6}$$

If, in addition, y_1 and y_2 are a fundamental set of solutions to (16.6), then

$$Y_1(x) - Y_2(x) = c_1y_1(x) + c_2y_2(x) \quad . \tag{16.7}$$

THEOREM 16.2. *Given one solution $y_p(x)$ of the nonhomogeneous differential equation*

$$y'' + p(x)y' + q(x)y = g(x) \tag{16.8}$$

then any other solution of this equation can be expressed as

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) \tag{16.9}$$

where y_1 and y_2 are two linearly independent solutions of the corresponding homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0 \quad . \tag{16.10}$$

Thus, to determine the general solution of a non-homogeneous linear equation (16.4), we can procede in three steps.

(1) Determine the general solution $c_1y_1(x) + c_2y_2(x)$ of the corresponding homogeneous problem.
(2) Find a particular solution $y_p(x)$ of the nonhomogeneous differential equation (16.6).
(3) Construct the general solution of (16.4) by setting

$$y(x) = y_p(x) + c_1y_1(x) + c_2y_2(x) \quad . \tag{16.11}$$

EXAMPLE 16.3. Given that one solution of

$$y'' + 3y' + 2y = e^{-x} \tag{16.12}$$

is $y(x) = xe^{-x}$, write down the general solution.

To construct the general solution we just apply the preceding theorem. We can immediately identify the function $y_p(x)$ in the theorem statement with our given solution xe^{-x}. To write down the general solution we also need two linearly independent solutions of

$$y'' + 3y' + 2 = 0\,. \tag{16.13}$$

Luckily, this is an equation we can solve. The characteristic equation for this homogeneous second order linear equation with constant coefficients is

$$\lambda^2 + 3\lambda + 2 = 0$$

which factors as

$$(\lambda + 2)(\lambda + 1) = 0$$

Thus, we have two real roots $\lambda = -2, -1$ and hence two linearly solutions of (16.13)

$$\begin{aligned} y_1(x) &= e^{-2x} \\ y_2(x) &= e^{-x} \end{aligned}$$

We now have all the ingredients we need to write down the general solution of (16.12):

$$\begin{aligned} y(x) &= y_p(x) + c_1 y_1(x) + c_2 y_2(x) \\ &= x e^{-x} + c_1 e^{-2x} + c_2 e^{-x} \end{aligned}$$