LECTURE 17

Variation of Parameters

Consider the differential equation

$$y'' + p(x)y' + q(x)y = g(x) \tag{17.1}$$

Suppose $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the homogeneous problem corresponding to (17.1); i.e., y_1 and y_2 satisfy

$$y'' + p(x)y' + q(x)y = 0 \tag{17.2}$$

and

$$W[y_1, y_2] \neq 0 \quad . \tag{17.3}$$

We seek to determine two functions $u_1(x)$ and $u_2(x)$ such that

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{17.4}$$

is a solution of (17.1). To determine the two functions u_1 and u_2 uniquely we need to impose two (independent) conditions. First, we shall require (17.4) to be a solution of (17.1); and second, we shall require

$$u_1'y_1 + u_2'y_2 = 0 \quad . \tag{17.5}$$

(This latter condition is imposed not only because we need a second equation, but also to make the calculation a lot easier.)

Differentiating (17.4) yields

$$y_p' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \tag{17.6}$$

which because of (17.5) becomes

$$y_p' = u_1y_1' + u_2y_2' \quad . \tag{17.7}$$

Differentiating again yields

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'' \quad . \tag{17.8}$$

We now plug (17.4), (17.7), and (17.8) into the original differential equation (17.1).

$$\begin{aligned} g(x) &= (u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'') + p(x)\,(u_1y_1' + u_2y_2') + q(x)\,(u_1y_1 + u_2y_2) \\ &= u_1'y_1' + u_2'y_2' + u_1\,(y_1'' + p(x)y_1' + q(x)y_1) + u_2\,(y_2'' + p(x)y_2' + q(x)y_2) \end{aligned} \tag{17.9}$$

The last two terms vanish since y_1 and y_2 are solutions of (17.2). We thus have

$$u_1'y_1 + u_2'y_2 = 0 \tag{17.10}$$

$$u_1'y_1' + u_2'y_2' = g \tag{17.11}$$

We now can now solve this pair of equations for u_1 and u_2. Rather than explicitly carry out the algebraic solution of equations (**??**) and (**??**), we'll use the following general fact:

Fact 17.1. *Let*

$$\begin{aligned} Ax + By &= e \\ Cx + Dy &= f \end{aligned}$$

be a pair of independent linear equations in two unknowns x and y. Then the solution of this system is given by

$$\begin{aligned} x &= \frac{eD - Bf}{AD - BC} \\ y &= \frac{Af - eC}{AD - BC} \end{aligned}$$

Thus, in the situation at hand, regarding (**??**) and (**??**) as a pair of linear equations for u_1' and u_2', we have

$$\begin{aligned} u_1' &= \frac{-y_2 g}{y_1 y_2' - y_1' y_2} = \frac{-y_2 g}{W[y_1, y_2]} \\ u_2' &= \frac{y_1 g}{y_1 y_2' - y_1' y_2} = \frac{y_1 g}{W[y_1, y_2]} \quad . \end{aligned} \tag{17.12}$$

(Note that division by $W(y_1, y_2)$ causes no problems since y_1 and y_2 were chosen such that $W(y_1, y_2) \neq 0$.) Hence

$$\begin{aligned} u_1(x) &= \int^x \frac{-y_2(t)g(t)}{W[y_1,y_2](t)} dt \\ u_2(x) &= \int^x \frac{y_1(t)g(t)}{W[y_1,y_2](t)} dx' \end{aligned} \tag{17.13}$$

and so

$$\boxed{y_p(x) = -y_1(x) \int^x \frac{y_2(t)g(t)}{W[y_1, y_2](t)} dt + y_2(x) \int^x \frac{y_1(t)g(t)}{W[y_1, y_2](t)} dt} \tag{17.14}$$

is a particular solution of (17.1).

Example 17.2. Find the general solution of

$$y'' - y' - 2y = 2e^{-x} \tag{17.15}$$

using the method of Variation of Parameters.

Well, the corresponding homogeneous problem is

$$y'' - y' - 2y = 0 \quad . \tag{17.16}$$

This is a second order linear equation with constant coefficients whose characteristic equation is

$$\lambda^2 - \lambda - 2 = 0 \quad . \tag{17.17}$$

The characteristic equation has two distinct real roots

$$\lambda = -1, 2 \tag{17.18}$$

and so the functions

$$\begin{aligned} y_1(x) &= e^{-x} \\ y_2(x) &= e^{2x} \end{aligned} \tag{17.19}$$

form a fundamental set of solutions to (17.16).

To find a particular solution to (17.15) we employ the formula (17.14). Now

$$g(x) = 2e^{-x} \tag{17.20}$$

and

$$W[y_1, y_2](x) = \left(e^{-x}\right)\left(2e^{2x}\right) - \left(-e^{-x}\right)\left(e^{2x}\right) = 3e^{x} \quad , \tag{17.21}$$

so

$$\begin{aligned} y_p(x) &= -y_1(x) \int^x \frac{y_2(t)g(t)}{W[y_1,y_2](t)} dt \quad + \quad y_2(x) \int^x \frac{y_1(t)g(t)}{W[y_1,y_2](t)} dt \\ &= -e^{-x} \int^x \frac{e^{2t}\left(2e^{-t}\right)}{3e^t} dt \quad + \quad e^{2x} \int^x \frac{e^{-t}\left(2e^{-t}\right)}{3e^t} dt \\ &= -e^{-x} \int^x \tfrac{2}{3} dt \quad + \quad e^{2x} \int^x \tfrac{2}{3} e^{-3t} dt \\ &= -\tfrac{2}{3} x e^{-x} - \tfrac{2}{9} e^{-x} \end{aligned} \tag{17.22}$$

The general solution of (17.15) is thus

$$
\begin{aligned}
y(x) &= y_p(x) + c_1 y_1(x) + c_2(x) \\
&= -\tfrac{2}{3}xe^{-x} + \left(c_1 - \tfrac{2}{9}\right)e^{-x} + c_2 e^{2x} \\
&= -\tfrac{2}{3}xe^{-x} + C_1 e^{-x} + C_2 e^{2x}
\end{aligned}
\tag{17.23}
$$

where we have absorbed the $-\frac{2}{9}$ in the second line into the arbitrary parameter C_1.