LECTURE 19

Application of Laplace Transforms to Initial Value Problems

Consider the following initial value problem.

$$y'' - y' - 2y = 0 \quad , \quad y(0) = 1 \quad , \quad y'(0) = 0 \quad . \tag{19.1}$$

One method of solving this initial value problem would be to use the techniques of Section 3.5.

We will develop here another method based on the Laplace transform.

Suppose $\mathcal{L}(y)$ is the Laplace transform of the solution of (19.1). Then according to the previous theorem, it must satisfy

$$0 = \mathcal{L}\left[y'' - y' - 2y\right] = s^2\mathcal{L}[y] - sy(0) - y'(0) - (s\mathcal{L}[y] - y(0)) - 2\mathcal{L}(y) \tag{19.2}$$

or

$$\begin{aligned} 0 &= \left(s^2 - s - 2\right)\mathcal{L}[y] - (s-1)y(0) - y'(0) \\ &= \left(s^2 - s - 2\right)\mathcal{L}[y] - (s-1) \end{aligned} \tag{19.3}$$

or

$$\mathcal{L}[y] = \frac{s-1}{s^2 - s - 2} = \frac{s-1}{(s+1)(s-2)} \quad . \tag{19.4}$$

We therefore look for a function whose Laplace transform is

$$\frac{s-1}{(s+1)(s-2)} \quad . \tag{19.5}$$

Now the above expression can be expanded using partial fractions as

$$\frac{s-1}{(s+1)(s-2)} = \frac{2}{3(s+1)} + \frac{1}{3(s-2)} \quad . \tag{19.6}$$

In Example 3. we found that

$$\mathcal{L}\left[e^{bx}\right] = \frac{1}{s-b} \qquad , \qquad s > b \quad . \tag{19.7}$$

Thus,

$$\begin{aligned} \mathcal{L}[y] &= \tfrac{s-1}{(s+1)(s-2)} \\ &= \tfrac{2}{3(s+1)} + \tfrac{1}{3(s-2)} \\ &= \tfrac{2}{3}\tfrac{1}{(s-(-1))} + \tfrac{1}{3}\tfrac{1}{(s-2)} \\ &= \tfrac{2}{3}\mathcal{L}\left[e^{-x}\right] + \tfrac{1}{3}\mathcal{L}\left[e^{2x}\right] \\ &= \mathcal{L}\left[\tfrac{2}{3}e^{-x} + \tfrac{1}{3}e^{3x}\right] \end{aligned} \tag{19.8}$$

So we should take

$$y(x) = \frac{2}{3}e^{-x} + \frac{1}{3}e^{3x} \quad . \tag{19.9}$$

Below I summarize how Laplace transforms can be used to solve ordinary differential equations.

1. Take the Laplace transform of both sides of the differential equation, using the identities

$$\begin{aligned} \mathcal{L}\left[y'\right](s) &= s\mathcal{L}[y] - y(0) \\ \mathcal{L}\left[y''\right](s) &= s^2\mathcal{L}[y] - sy(0) - y'(0) \end{aligned} \tag{19.10}$$

to replace the Laplace transforms of the derivative terms. Be sure to replace $y(0)$ and $y'(0)$ by the appropriate values (from the initial conditions).

2. Solve (algebraically) the resulting equation in order to express $\mathcal{L}[y]$ as an explicit function of s.

3. Try to identify a function f(x) such that $\mathcal{L}[f](s)$ is the function $\mathcal{L}[y]$ of s found in Step 2. To figure out f, first consult the table of Laplace transforms to see if you can recognize $\mathcal{L}[y]$ as the Laplace transform of some elementary function. If nothing pops out, try applying partial fractions to reduce a (more complicated) rational function of s to a sum of simpler functions, and then consult the Laplace transform tables again.

4. The solution of the differential equation will be the function $f(x)$ determined in Part 3.

1. Review of Partial Fractions

Partial fraction expansions are a very useful tool for figuring out inverse Laplace transforms. The basic utility of the partial fraction expansion is that it can be used to replace a complicated rational function (a ratio of two polynomials) by a sum of simpler rational functions. Suppose $f(x)$ is rational function:

$$f(x) = \frac{p_n x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0}{q_m x^m + q_{m-1}x^{m-1} + \cdots + q_1 x + q_0}$$

i.e. a ratio of two polynomials in x. We shall assume that we have already simplified this expression for $f(x)$ such that the numerator and denominator have no common factors. If the degree of the numerator is greater than the degree of the denominator, then we can use polynomial division to reduce $f(x)$ to the sum of a polynomial in x plus a rational function as a remainder. We shall assume that the simplification of $f(x)$ has already proceded this far and we are now concentrating on the remainder, which would be of the form

$$f(x) = \frac{p_{m-1}x^{m-1} + p_{m-2}x^{m-2} + \cdots + p_1 x + p_0}{q_m x^m + q_{m-1}x^{m-1} + \cdots + q_1 x + q_0}$$

According to the Fundamental Theorem of Algebra, the denominator can be written as a product of irreducible linear and quadratic factors

$$q_m x^m + q_{m-1}x^{m-1} + \cdots + q_1 x + q_0 = q_m (x - a_i)^{m_1} \cdots (x - a_s)^{m_s} (x^2 + b_1 x + c_1)^{n_1} \cdots (x^2 + b_t x + c_t)^{n_t}$$

with each $a_i, b_j, c_k \in \mathbb{R}$. (Note here that we're factoring over the real numbers; that's why we cannot assume that we have a complete factorization in terms of factors linear in x.)The partial fractions expansion of $R(x)$ is a expression of the form

$$\begin{aligned} R(x) \quad = \quad & \frac{A_{1,1}}{x - a_1} + \frac{A_{1,2}}{(x - a_1)^2} + \cdots + \frac{A_{1,m_1}}{(x - a_1)^{m_1}} + \cdots \\ & \cdots + \frac{A_{s,1}}{x - a_s} + \frac{A_{s,2}}{(x - a_s)^2} + \cdots + \frac{A_{s,m_s}}{(x - a_s)^{m_s}} \\ & + \frac{B_{1,1}x + C_{1,1}}{(x^2 + b_1 x + c_1)} + \frac{B_{1,2}x + C_{1,2}}{(x^2 + b_1 x + c_1)^2} + \cdots + \frac{B_{1,n_1}x + C_{1,n_1}}{(x^2 + b_1 x + c_1)^{n_1}} + \cdots \\ & \cdots + \frac{B_{t,1}x + C_{t,1}}{(x^2 + b_t x + c_t)} + \frac{B_{t,2}x + C_{t,2}}{(x^2 + b_t x + c_t)^2} + \cdots + \frac{B_{t,n_t}x + C_{t,n_t}}{(x^2 + b_1 x + c_1)^{n_t}} \end{aligned}$$

This formula can be summarized very easily as follows: corresponding to each factor in the denominator of the form $(x - a)^m$ we have m terms in the partial fractions expansion:

$$\frac{1}{(x - a)^m} \qquad \rightsquigarrow \qquad \frac{A_1}{x - 1} + \frac{A_2}{(x - 1)^2} + \cdots + \frac{A_m}{(x - 1)^m}$$

and corresponding to each factor of the form $\left(x^2+bx+c\right)^m$ in the denominator we have m terms in the partial fractions expansion

$$\frac{1}{(x+bx+c)^m} \quad \rightsquigarrow \quad \frac{B_1x+C_1}{x+bx+c}+\frac{B_2x+C_2}{(x+bx+c)^2}+\cdots+\frac{B_mx+C_m}{(x+bx+c)^m}$$

2. Hints for Inverting Laplace Transforms

Suppose you need to find the function $y(x)$ such that

$$\mathcal{L}[y]=F(s)=\frac{P(s)}{Q(s)}$$

with $P(s)$ and $Q(s)$ polynomials.

2.1. Case 1: $Q(s)$ factorizes completely.

If $Q(s)$ can be easily factorized in terms of linear factors, replace $F(s)$ by its partial fractions expansion, identify the inverse Laplace transform of each term in the partial fractions expansion, and then sum the resulting inverse Laplace transforms up to get $y(x)$.

- Here is a quick reminder of how Partial Fractions works: Suppose

$$F(s)=\frac{P(s)}{(s-a)^{m_1}(s-b)^{m_2}\cdots(s-c)^{m_3}}$$

with the polynomial degree of $P(s) <$ the polynomial degree of the denominator. Then there exists constants $A_1, A_2, \ldots, A_{m_1}, B_1, B_2, \ldots, B_{m_2}, \cdots, C_1, \ldots, C_{m_3}$ such that

$$\begin{aligned} F(s) &= \frac{A_1}{(s-a)}+\frac{A_2}{(s-a)^2}+\cdots+\frac{A_{m_1}}{(s-a)^{m_1}} \\ &\quad +\frac{B_1}{(s-b)}+\frac{B_2}{(s-b)^2}+\cdots+\frac{B_{m_2}}{(s-b)^{m_2}} \\ &\quad \vdots \\ &\quad +\frac{C_1}{(s-c)}+\frac{C_2}{(s-c)^2}+\cdots+\frac{C_{m_3}}{(s-c)^{m_3}} \end{aligned}$$

In practice, it is just a bit of tedius algebra to find the correct choice of the constants $A_1, \ldots, C_{m_3}$. However, once they have been determined the Laplace transform of $F(s)$ is completely determined: since

$$\mathcal{L}\left[x^n e^{ax}\right]=\frac{n!}{(s-a)^{n+1}} \quad \Rightarrow \quad \frac{1}{(s-a)^n}=\mathcal{L}\left[\frac{1}{(n-1)!}x^{n-1}e^{ax}\right]$$

we'll have

$$\begin{aligned} F(s) &= A_1\mathcal{L}\left[\frac{1}{0!}e^{ax}\right]+A_2\mathcal{L}\left[\frac{1}{1!}xe^{ax}\right]+\cdots+A_{m_1}\mathcal{L}\left[\frac{1}{(m_1-1)!}x^{m_1-1}e^{ax}\right] \\ &\quad \vdots \\ &\quad C_1\mathcal{L}\left[\frac{1}{0!}e^{cx}\right]+C_2\mathcal{L}\left[\frac{1}{1!}xe^{cx}\right]+\cdots+C_{m_3}\mathcal{L}\left[\frac{1}{(m_1-1)!}x^{m_3-1}e^{cx}\right] \\ &= \mathcal{L}\left[A_1e^{ax}+A_2xe^{ax}+\cdots C_{m_3}x^{m_3-1}e^{cx}\right] \end{aligned}$$

Now in most of the examples you'll see in the homework, $Q(x)$ is a polynomial of degree two; and so is either a product of the form $(s-a)(s-b)$ or a perfect square $(s-a)^2$. Thus, for the most part, you'll only need the following two partial fraction expansions

$$F(s) = \frac{P(s)}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b} = \mathcal{L}\left[Ae^{ax} + Be^{bx}\right]$$

or

$$F(s) = \frac{P(s)}{(s-a)^2} = \frac{A}{s-a} + \frac{B}{(s-a)^2} = \mathcal{L}\left[Ae^{ax} + Bxe^{ax}\right]$$

2.2. Case Two: $Q(s)$ does not factor easily. In this case, the approach to take is to try to complete the square of denominator.

Suppose

$$F(s) = \frac{P(s)}{s^2 + Bs + C}$$

Then since

$$\begin{aligned} s^2 + Bs + c &= s^2 + Bs + \left(\frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^2 + C \\ &= \left(s + \frac{B}{2}\right)^2 + \left(C - \frac{B^2}{4}\right) \end{aligned}$$

Here there are two subcases

- If $\left(C - \frac{B^2}{4}\right)$ is positive, we can take its square root, and so

  $$s^2 + Bs + C = (s-a)^2 + b^2 \qquad \text{with } a = -\frac{B}{2} \text{ and } b = \sqrt{C - B^2/4}$$

  Thus,

  $$F(s) = \frac{P(s)}{(s-a)^2 + b^2}$$

  We can then try to find constants α and β such that

  $$\begin{aligned} F(s) &= \alpha\frac{(s-a)}{(s-a)^2 + b^2} + \beta\frac{b}{(s-a)^2 + b^2} \\ &= \alpha\mathcal{L}\left[e^{ax}\cos(bs)\right] + \beta\mathcal{L}\left[e^{ax}\sin(bx)\right] \\ &= \mathcal{L}\left[\alpha e^{ax}\cos(bs) + \beta e^{ax}\sin(bx)\right] \end{aligned}$$

  and so

  $$\mathcal{L}^{-1}(F(s)) = \alpha e^{ax}\cos(bs) + \beta e^{ax}\sin(bx)$$

- If $\left(C - \frac{B^2}{4}\right)$ is negative, we can take the square root of $B^2/4 - C$, and so

  $$s^2 + Bs + C = (s-a)^2 - b^2 \qquad \text{with } a = -\frac{B}{2} \text{ and } b = \sqrt{B^2/4 - C}$$

  Thus,

  $$F(s) = \frac{P(s)}{(s-a)^2 - b^2}$$

  We can then try to find constants α and β such that

  $$\begin{aligned} F(s) &= \alpha\frac{(s-a)}{(s-a)^2 - b^2} + \beta\frac{b}{(s-a)^2 - b^2} \\ &= \alpha\mathcal{L}\left[e^{ax}\cosh(bs)\right] + \beta\mathcal{L}\left[e^{ax}\sinh(bx)\right] \\ &= \mathcal{L}\left[\alpha e^{ax}\cosh(bs) + \beta e^{ax}\sinh(bx)\right] \end{aligned}$$

  and so

  $$\mathcal{L}^{-1}\left[F(s)\right] = \left[\alpha e^{ax}\cosh(bs) + \beta e^{ax}\sinh(bx)\right]$$