LECTURE 20

Laplace Transforms and Piecewise Continuous Functions

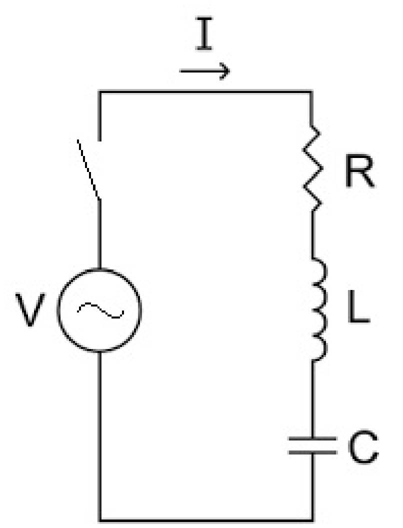
We have seen how one can use Laplace transform methods to solve 2^{nd} order linear Diff E's with constant coefficients, and have even pointed out some advantages of the Laplace transform technique over our original method of solving inhomogenous boundary value problems (where we first solved the characteristic equation to find two independent solutions y_1 and y_2 of the corresponding homogenous equation, then used Variation of Parameters to get a particular solution y_p of the inhomogenenous equation, and finally plugged $y = y_p + c_1y_1 + c_2y_2$ into the initial conditions to obtain the correct choice of c_1 and c_2).

Another big advantage is that the Laplace transform technique allows us to solve Diff E's of the form

$$ay'' + by' + cy = g(x)$$

where $g(x)$ is only a piecewise continuous function.

Here's one example of how such a differential equation might arise. Consider a simple LRC circuit being driven by the current in a wall outlet. The circuit diagram looks like



where V represents the time dependent voltage produced at the wall socket (something like $V(t) = 110V \sin\left(\frac{\pi}{30}t\right)$). Transversing the circuit and summing up the voltages difference incurred at each component lead to the following differential equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t) \quad .$$

However, in actual practice the circuit is turned on at some time, say $t = 0$, and accounting for the transient effects that take effect when the circuit gets turned on are an important engineering consideration.

In particular, in *practice* the driving voltage function $V(t)$ is actually a discontinuous function more like

$$V(t) = \begin{cases} 110\sin\left(\frac{\pi}{30}t\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

In this lecture, I'll show you how to solve such differential equations with discontinuous driving function via the Laplace Transform Method.

1. Modeling Piecewise Continuous Fucntions with Heaviside Functions

Suppose you had a piecewise continuous function of the form

$$f(t) = \begin{cases} 0 & t < 0 \\ t^2 & t \geq 0 \end{cases} \tag{1}$$

The mathematician Oliver Heaviside introduced the following simple function to ease the manipulation of discontinuous functions:

$$u(t) \equiv \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

With the function we can reexpress the function $f(t)$ given in (1) as

$$f(t) = t^2 u(t)$$

effectively using the factor $u(t)$ to account for the discontinuity in $f(t)$ at $t = 0$ and using t^2 to model the behavior of $f(t)$ after $t \geq 0$.

Oftentimes one deals with functions that get turned on and off. For example,

$$f(t) = \begin{cases} 0 & \text{if } t < a \\ \sin(t) & \text{if } a \leq t \leq b \\ 0 & \text{if } t > b \end{cases}$$

can be expressed as

$$f(t) = u(t-a)\sin(t) - u(t-b)\sin(t) = (u(t-a) - u(t-b))\sin(t)$$

Note how that the $u(t-a)\sin(t)$ *turns on* the function $\sin(t)$ when $t = a$, and then when t reaches b the term $-u(t-b)\sin(t)$ gets turned on to cancel the contribution from $u(t-a)\sin(t)$.

NOTATION 20.1. Below I'll sometimes write $u_a(t)$ for the function $u(t-a)$:

$$u_a(t) \equiv u(t-a) \quad .$$

2. Laplace Transforms of Piecewise Continuous Functions

THEOREM 20.2. *Suppose $f(t)$ has a Laplace transform $\mathcal{L}[f](s)$. Then*

$$\mathcal{L}[u(t-a) f(t-a)] = e^{-as}\mathcal{L}[f](s)$$

Proof.

$$\mathcal{L}[u(t-a) f(t-a)] \equiv \int_0^\infty u(t-a) f(t-a) e^{-st} dt$$

Making a change of variables $t \to t' = t - a$

$$\begin{aligned}
&= \int_{-a}^{\infty} u(t') f(t') e^{-s(t'+a)} dt' \\
&= \int_{0}^{\infty} f(t') e^{-st'} e^{-sa} dt' \qquad \text{since } u(t') = 0 \text{ for } t < 0 \\
&= e^{-sa} \int_{0}^{\infty} f(t') e^{-st'} dt' \\
&= e^{-sa} \mathcal{L}[f](s)
\end{aligned}$$

The formula in the theorem is not quite the formula one first applies in practice. To compute the Laplace transform of a discontinuous function like

$$g(t) = \begin{cases} 0 & t < a \\ f(t) & t \geq a \end{cases}$$

we'd want take the Laplace transform of $u(t-a) f(t)$, rather than the function $u(t-a) f(t-a)$ like in statement of Theorem 20.2. To handle this discrepancy, one defines a new shifted function $f_a(t)$ by

$$f_a(t-a) = f(t) \quad \Longleftrightarrow \quad f_a(t) = f(t+a)$$

Then we have

$$\mathcal{L}[u(t-a) f(t)] = \mathcal{L}[u(t-a) f_a(t-a)] = e^{-as} \mathcal{L}[f_a(t)]$$

I'll now restate the Theorem in the two common ways it is applied in practice:

COROLLARY 20.3. (i) *Let*

$$f_a(t) \equiv f(t+a)$$

Then

$$\mathcal{L}[u(t-a) f(t)] = e^{-as} \mathcal{L}[f_a(t)]$$

(ii) *Suppose the inverse Laplace transform of $F(s)$ is $f(t)$. Then*

$$\mathcal{L}^{-1}\left(e^{-as} F(s)\right) = u(t-a) f(t-a) \quad .$$

EXAMPLE 20.4. Compute the Laplace transform of the following discontinous function

$$g(t) = \begin{cases} 0 & t < 2 \\ t^2 & t \geq 2 \end{cases}$$

- First we reconstruct $g(t)$ using the Heaviside function $u(t-2)$

  $$g(t) = u(t-2) t^2$$

  Thinking of this as

  $$g(t) = u(t-a) f(t)$$

  we have

  $$a = 2 \quad , \quad f(t) = t^2$$

  This makes $f_a(t)$ in the Corollary the function

  $$f_a(t) = f_2(a) = f(t+2) = (t+2)^2 = t^2 + 4t + 4$$

  Thus, by part (i) of the Corollary,

  $$\begin{aligned}
  \mathcal{L}[g(t)] &= \mathcal{L}[u(t-2) f_2(t)] = e^{-2s} \mathcal{L}[f_2(t)] = e^{-2s} \mathcal{L}\left[t^2 + 4t + 4\right] \\
  &= e^{-2s} \left(\mathcal{L}\left[t^2\right] + 4\mathcal{L}[t] + \mathcal{L}[4]\right)
  \end{aligned}$$

  From a Table of Laplace transforms one finds

  $$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

and so

$$\begin{aligned} \mathcal{L}\left[g\left(t\right)\right] &= e^{-2s}\left(\frac{2}{s^3}+4\frac{1}{s^2}+4\frac{1}{s}\right) \\ &= e^{-2s}\left(\frac{2}{s^3}+\frac{4}{s^2}+\frac{4}{s}\right) \end{aligned}$$

EXAMPLE 20.5. Compute the inverse Laplace transform of

$$G\left(s\right)=e^{-2s}\frac{1}{s+3}$$

Since

$$\mathcal{L}\left[e^{-3t}\right]=\frac{1}{s+3}$$

$G\left(s\right)$ is of the form

$$e^{-2s}\mathcal{L}\left[e^{-3x}\right]$$

and so we can apply part (ii) of the Corollary, using $a=2$ and $f\left(t\right)=e^{-3t}$. Thus

$$\mathcal{L}^{-1}\left[e^{-2s}\frac{1}{s+3}\right]=u\left(t-2\right)e^{-3(t-2)}$$

3. Solving Differential Equations with Discontinous Driving Functions

EXAMPLE 20.6. Find the solution of

$$\begin{aligned} y''+y &= f\left(t\right)=\left\{\begin{array}{lll} 1 & , & 0\leq t\leq\frac{\pi}{2} \\ 0 & , & \frac{\pi}{2}<t<\infty \end{array}\right. \\ y\left(0\right) &= 0 \\ y'\left(0\right) &= 0 \end{aligned}$$

(This might correspond to a simple harmonic oscillator that was initially jolted by a constant force for $\frac{\pi}{2}$ seconds, and left alone.)

Notice that the driving function $f\left(t\right)$ is just

$$f\left(t\right)=u_0\left(t\right)-u_{\pi/2}\left(t\right)$$

Now, recalling the formula for the Laplace transform

$$\mathcal{L}\left[f\right]=\int_0^\infty f\left(t\right)e^{-st}dt$$

only uses the values of f for when $t\geq 0$, we may as well set $u_0\left(t\right)=1$ (since $u_0\left(t\right)$ has value 1 for all $t>0$). Hence the Laplace transform of $f\left(t\right)$ will be

$$\mathcal{L}\left[f\right]=\mathcal{L}\left[1\right]-\mathcal{L}\left[u_{\pi/2}\right]=\frac{1}{s}-\frac{e^{-\frac{\pi s}{2}}}{s}=\frac{1}{s}\left(1-e^{-\frac{\pi s}{2}}\right)$$

So the Laplace transform of the differential equation will be

$$s^2\mathcal{L}\left[y\right]-sy\left(0\right)-y'\left(0\right)+\mathcal{L}\left[y\right]=\frac{1}{s}\left(1-e^{-\frac{\pi s}{2}}\right)$$

or

$$\left(s^2+1\right)\mathcal{L}\left[y\right]=\frac{1}{s}\left(1-e^{-\frac{\pi s}{2}}\right)$$

or

$$\begin{aligned} \mathcal{L}\left[y\right] &= \frac{1}{s\left(s^2+1\right)}\left(1-e^{-\frac{\pi s}{2}}\right) \\ &= \frac{1}{s\left(s^2+1\right)}-\frac{e^{-\frac{\pi s}{2}}}{s\left(s^2+1\right)} \end{aligned}$$

Now

$$\frac{1}{s\left(s^2+1\right)}=\frac{A}{s}+\frac{Bs+C}{s^2+1} \quad \Rightarrow \quad 1=A\left(s^2+1\right)+Bs^2+Cs \quad \Rightarrow \quad \left\{\begin{array}{l} A=1 \\ B=-1 \\ C=0 \end{array}\right.$$

So

$$\frac{1}{s\left(s^2+1\right)}=\frac{1}{s}-\frac{s}{s^2+1}=\mathcal{L}\left[1\right]+\mathcal{L}\left[\cos\left(t\right)\right]=\mathcal{L}\left[1+\cos\left(t\right)\right]$$

or

$$\mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]=1+\cos\left(t\right)$$

On the other hand, according to part (ii) of the Corollary, if $\mathcal{L}\left[f\right]=F\left(s\right)$, then

$$\mathcal{L}^{-1}\left[e^{-as}F\left(s\right)\right]=u\left(x-a\right)f\left(x-a\right)$$

so

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi s}{2}}}{s\left(s^2+1\right)}\right] &= u_{\pi/2}\left(t\right)\mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]\Bigg|_{t-\pi/2}=u_{\pi/2}\left(t\right)\left(1+\cos\left(t-\pi/2\right)\right) \\
&= u\left(t-\frac{\pi}{2}\right)\left(1-\sin\left(t\right)\right)
\end{aligned}$$

(since $\cos\left(t-\frac{\pi}{2}\right)=-\sin\left(t\right)$). Thus,

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left[\mathcal{L}\left[y\right]\right] \\
&= \mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}-\frac{e^{-\frac{\pi s}{2}}}{s\left(s^2+1\right)}\right] \\
&= \mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right]-\mathcal{L}^{-1}\left[\frac{e^{-\frac{\pi s}{2}}}{s\left(s^2+1\right)}\right] \\
&= 1+\cos\left(t\right)-u_{\pi/2}\left(t\right)\left(1-\sin\left(t\right)\right)
\end{aligned}$$

4. Impulse Functions - The Dirac Delta Function

We have seen the Laplace transform technique is very good for solving differential equations

$$ay''+by'+cy=g\left(x\right)$$

when the "driving function" $g\left(s\right)$ is only piecewise continuous. Physically such a differential equation might arise if an oscillatory system were given an initial push, or a recurrent push. But what happens when an oscillatory system is struck by a hammer?

To discuss such situations we first need a measure of how much energy is transferred to the system after the application of a constant force. As a crude measure of the amount of energy imparted to a system driven by a force $g(t)$ we introduce the **total impulse** I_g defined by

$$I_g\equiv\int_{-\infty}^{+\infty}g(t)dt$$

The way to think about this quantity is as follows: if $g(t)$ corresponds to the force applied to the system at time t, then I_g is the aggregate force applied to system over all time. Note how the magnitude of I_g depends not only on the magnitude of $g\left(t\right)$, but also on how long the force was applied (i.e. how long $g\left(t\right)$ is non-zero).

Now consider the following four driving functions

$$\begin{aligned}
g_1(t) &= \begin{cases} \frac{1}{4} & , \quad -2 \le t \le 2 \\ 0 & , \quad |t| > 2 \end{cases} & \Rightarrow \quad I_{g_1} = 1 \\
g_2(t) &= \begin{cases} \frac{1}{2} & , \quad -1 \le t \le 1 \\ 0 & , \quad |t| > 1 \end{cases} & \Rightarrow \quad I_{g_2} = 1 \\
g_3(t) &= \begin{cases} 1 & , \quad -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & , \quad |t| > \frac{1}{2} \end{cases} & \Rightarrow \quad I_{g_3} = 1 \\
g_4(t) &= \begin{cases} 2 & , \quad -\frac{1}{4} \le t \le \frac{1}{4} \\ 0 & , \quad |t| > \frac{1}{4} \end{cases} & \Rightarrow \quad I_{g_4} = 1
\end{aligned}$$

Note how the total impulse are all the same. More generally, if we set

$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau} & , \quad -\tau \le t \le \tau \\ 0 & , \quad |t| > \tau \end{cases}$$

Then

$$I_{d_\tau} = \int_{-\infty}^{+\infty} d_\tau(t)\,dt = \int_{-\tau}^{\tau} \frac{1}{2\tau} dt = \left.\frac{t}{2\tau}\right|_{-\tau}^{\tau} = \frac{1}{2\tau}(\tau - (-\tau)) = \frac{2\tau}{2\tau} = 1$$

So all the driving functions $d_\tau(t)$, $\tau \in \mathbb{R}^+$ deliver the same total impulse, $I_{d_\tau} = 1$. We use this sort of driving functions to model situations like a hammer strike. For in such situations we want the duration of the force to be nearly instantaneous, yet we want a finite amount of energy to be transferred to the system. In fact, the situation we would really like to handle is the case where all the energy is transferred at a single instant $t = 0$. For this we would need something like

$$g(t) = \lim_{\tau \to 0} d_\tau(t)$$

However, there is no limit to $d_\tau(t)$ as $\tau \to 0$, since $\lim_{\tau \to 0} d_\tau(0) = \lim_{\tau \to 0} \frac{1}{2\tau} = \pm\infty$. Evidently, the family of unit impulse functions functions $d_\tau(t)$, $\tau > 0$, fails to converge to a function as $\tau \to 0$.

The surprising fact (at least at first) is that even though

$$\lim_{\tau \to 0} d_\tau$$

does not exist, its integral from $-\infty$ to $+\infty$ does: because

$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t)\,dt = \lim_{\tau \to 0} 1 = 1$$

In fact, if $f(x)$ is any continuous function on the real line

$$\lim_{\tau \to 0} \int_{-\infty}^{+\infty} d_\tau(t) f(t)\,dt = f(0)$$

In this sense, we define the **Dirac delta function** $\delta(t)$

$$\delta(t) = \lim_{\tau \to 0} d_\tau(t)$$

with the understanding that it is not really a proper function, but nevertheless it has the property that when integrated from $-\infty$ to $+\infty$ against any function $f(t)$ the result is $f(0)$:

$$\int_{-\infty}^{+\infty} \delta(t) f(t)\,dt = f(0)$$

Now here's another surprising fact, $\delta(t)$ is not a function, but nevertheless it can still be differentiated (at least formally), so long as we keep it inside an integral. To evaluate

$$\int_{-\infty}^{+\infty} \delta'(t) f(t) dt$$

we simply apply integration by parts; using the integration by parts formula $\int u dv = uv - \int v du$ and the identifications

$$\begin{aligned} u &= f(t) \quad , \qquad dv = \delta'(t)\,dt \\ du &= f'(t)\,dt \quad , \qquad v = \delta(t) \end{aligned}$$

we have

$$\int_{-\infty}^{+\infty} \delta'(t)\, f(t) dt = \left. f(t)\,\delta(t)\right|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} \delta(t)\, f'(t)\, dt$$

evaluation of $f(t)\,\delta(t)$ at the endpoints of integration yields 0 because $\delta(t) = 0$ for all $t \neq 0$, on the other hand, from the definition of $\delta(t)$

$$\int_{-\infty}^{+\infty} \delta(t)\, f'(t)\, dt = f'(0)$$

Thus,

$$\int_{-\infty}^{+\infty} \delta'(t)\, f(t) dt = -f'(0)\,.$$

and so $\delta'(t)$ is the (generalized) function that when integrated against a function $f(t)$, yields $-f'(0)$

Example 20.7. Solve the following initial value problem:

$$\begin{aligned} y'' + 2y' + 2y &= \delta(t-1) \\ y(0) &= 0 \\ y'(0) &= 0 \end{aligned}$$

(You can imagine this initial value problem as corresponding to a damped harmonic oscillator, initially at rest, and then struck by a hammer at time $t = 1$.

Taking the Laplace transform of the differential equation we get

$$s^2 \mathcal{L}[y] - sy(0) - y'(0) + 2\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] = \mathcal{L}[\delta(t-1)] = e^{-s}$$

or

$$\left(s^2 + 2s + 2\right) \mathcal{L}[y] = e^{-s} \quad \Rightarrow \quad \mathcal{L}[y] = \frac{e^{-s}}{s^2 + 2s + 2}$$

Now

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{(s+1)^2 + 1} = \mathcal{L}\left(e^{-t} \sin(t)\right)$$

Applying Theorem 31.3, viz,

$$\text{If } F(s) = \mathcal{L}[f] \quad \text{, then } \mathcal{L}^{-1}\left(e^{-cs} F(s)\right) = u_c(t)\, f(t-c)$$

we have

$$\begin{aligned} \mathcal{L}^{-1}\left(e^{-s} \frac{1}{(s+1)^2 + 1}\right) &= u_1(t)\, \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2 + 1}\right)(t-1) = u_1(t)\, e^{-(t-1)} \sin(t-1) \\ &= \begin{cases} 0 & \text{if } t < 1 \\ e^{-t+1} \sin(t-1) & \text{if t} \geq 1 \end{cases} \end{aligned}$$