LECTURE 21

Solutions via Power Series

1. Introduction

Before discussing power series it is instructive to first admit up front the ludicrous strategy we have employed thus far to find solutions to second order linear differential equations. At this point only there are only two classes of equations that we can solve

- second order linear with constant coefficients
- Euler type equations

and our procedure for constructing a solution has been to first guess what the solution should look like ($y(x) = e^{\lambda x}$ for linear equations with constant coefficients, and $y(x) = x^r$ for Euler type equations) and then plug in to the differential equation to find a choice of λ or r that actually makes our given guess a solution. Such a strategy surely cannot work in any generality, because in general we have no clue as to what the solution of a second order equation looks like.

But, in fact, we can employ this same strategy with great success; it's just that we have to be sufficiently general in guessing what a solution should look like.

Here's the basic idea in a nutshell. Every smooth function has a unique representation in terms of its Taylor series about $x = 0$:

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \qquad (1) \\ &= f(x_o) + f'(0)x + \frac{1}{2}f''(0)x + \frac{1}{6}f'''(0)x^3 + \cdots \end{aligned}$$

Think of this as an expresion for $f(x)$ of the form

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \cdots$$

If we can figure out a way choose the constant coefficients $a_0, a_1, a_2, a_3, \ldots$ so that a differential equation is satisfied, then we've found solution.

In fact, we've done this already for first order equations. Let me show you that the same ideas apply in the second order case.

EXAMPLE 21.1. Find the first five terms of the Taylor expansion about $x = 0$ of the solution to

$$\begin{aligned} y'' + 2xy' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

The Taylor expansion of the solution $y(x)$ about $x = 0$ is given by the formula

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \cdots$$

To make this explicit, we need to figure out numerical values for $y(0), y'(0), y''(0), \ldots$. Now the values of $y(0)$ and $y'(0)$ are detemined by the initial conditions

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

The differential equation itself, evaluated at $x = 0$ gives us the value of $y''(0)$:

$$y''(0) = (-2xy' - y)\,|_{x=0} = 0 - y(0) = -1$$

To get a value for $y'''(0)$ we differentiate the differential equation and evaluate the result at $x = 0$:

$$y'''(0) = (-2y'(x) - 2xy''(x) - y'(x))\,|_{x=0} = 0 - 0 - 0 = 0$$

To get a value for $y^{iv}(0)$ we differentiate the differential equation again:

$$y^{iv}(0) = \left(-2y''(x) - 2y''(x) - 2xy''^{\prime}(x) - y''(x)\right)|_{x=0} = -2(-1) - 2(-1) - 0 - (-1) = 5$$

Thus, to order x^4

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \cdots \\ &= 1 + 0x - \frac{1}{2}x^2 - \frac{0}{6}x^3 + \frac{5}{24}x^4 + \cdots \\ &= 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots \end{aligned}$$

□

Note that this Taylor series technique is exactly the same as the one we discussed for first order differential equations. What we shall be doing in the next couple of lectures is systematizing this procedure for the case of second order linear differential equations. In doing so, we will not only be able to write down the Taylor expansions of solutions satisfying given initial conditions, but also the Taylor expansions of general solutions as well.

Let's condense our notation a bit by setting

$$a_n = \frac{f^{(n)}(x_o)}{n!} \tag{2}$$

so that Taylor expansion can be expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \tag{3}$$

If we had a formula for $f(x)$ then obviously we could compute each of the coefficients a_n in its Taylor expansion using equation (2). On the other hand, if we have formulas for all the coefficients a_n then can still write down the Taylor expansion of $f(x)$ via (3) and so we have effectively determined $f(x)$. The point of all this is that every smooth function can be expressed in the form (3) and by determining all the values of the constants a_n you effectively specify $f(x)$.

Now I can state our strategy for solving a genereal second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

We shall assume that our solution is a smooth function and so it has a Taylor expansion about a given point x_o:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

We'll then plug this expression for $y(x)$ into the differential equation and try to determine what this implies about the coefficients a_n. What we'll find is that the differential equation will effectively determine all the coefficients a_n in terms of the first two; and that the first two coefficients, a_0 and a_1, are determined completely by intitial conditions at x_o. Thus, we **will** be able to solve second order linear equations in the sense that we can construct the Taylor series representations of their solutions.

Before we can undertake this program in ernest we had better first review the basic theory concerning expressiions of the form

$$\sum_{n=0}^{\infty} a_n (x - x_o)^n$$

1.1. Review of Power Series. Recall that a **formal power series about** x_o is a formal expression of the form

$$\sum_{n=0}^{\infty} a_n (x - x_o)^n = a_0 + a_1 (x - x_o) + a_2 (x - x_o)^2 + \cdots \quad . \tag{4}$$

The reason for the qualification *formal* is that an expression of this form (as it stands) really doesn't make any mathematical sense: there is no way one can actually carry out the infinite summation implied by the notation.

However, using the notion of *limits* one can sometimes prescribe some real mathematical meaning to formal power series.

DEFINITION 21.2. *A formal power series* $\sum_{n=0}^{\infty} a_n (x - x_o)^n$ ***converges*** *at a point* x *if the limit*

$$\lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_o)^n$$

exists.

DEFINITION 21.3. *A formal power series* $\sum_{n=0}^{\infty} a_n (x - x_o)^n$ *is said to* ***converge absolutely*** *if the limit*

$$\lim_{N \to \infty} \sum_{n=0}^{N} |a_n (x - x_o)^n|$$

exists.

We recall that absolute convergence implies convergence but that convergence does not necessarily guarantee absolute convergence.

THEOREM 21.4. *If a formal series* $\sum_{n=0}^{\infty} a_n (x - x_o)^n$ *converges for some* $x \neq 0$*, say* $x = x_1$*, then the series converges absolutely for all* x *such that*

$$|x - x_o| < |x - x_1| \, .$$

The largest number R such that a power series

$$\sum_{n=0}^{\infty} a_n (x - x_o)^n$$

converges for all $x \in (x_o - R, x_o + R)$ (or equivalently, for all x such that $|x - x_o| < R$) is called the **radius of convergence** of the power series.

The following test is very useful in determining whether a given power series converges.

THEOREM 21.5. ***(Ratio Test.)*** *A formal series* $\sum_{n=0}^{\infty} a_n (x - x_o)$ *converges absolutely if*

$$1 > \lim_{n \to \infty} \left| \frac{a_{n+1} (x - x_o)^{n+1}}{a_n (x - x_0)^n} \right| = |x - x_o| \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| \quad .$$

This test implies that the radius of convergence of $\sum_{n=0}^{\infty} a_n(x-x_o)^n$ is given by

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

Example 21.6. Find the radius of convergence of the following power series.

$$\sum_{n=0}^{\infty} \frac{n}{2^n}(x-1)^n \quad .$$

Well, we have $a_n = \frac{n}{2^n}$, so

$$R = \lim_{n\to\infty}\left|\frac{\frac{n}{2^n}}{\frac{n+1}{2^{n+1}}}\right| = \lim_{n\to\infty} 2\frac{n}{n+1} = 2\lim_{n\to\infty}\frac{n}{n+1} = 2 \quad .$$

We conclude that the series converges for all x such that $|x-1| < 2$, i.e., for all $x \in (-1,3)$.

Now if a power series $\sum_{n=0}^{\infty} a_n(x-x_o)^n$ has a radius of convergence R then for all $x \in (x_o - R, x_o + R)$ we have a well- defined function of x; viz.,

$$f(x) = \lim_{N\to\infty}\sum_{n=0}^{N} a_n(x-x_o)^n \quad .$$

This function is not only continuous within the interval $(x_o - R, x_o + R)$, all of its derivatives $f^{(n)}$ exist as well. In fact, the derivative of f is the function defined by

$$f'(x) = \lim_{N\to\infty}\sum_{n=0}^{N} n a_n(x-x_o)^{n-1}$$

which is also defined for all $x \in (x_o - R, x_o + R)$.

Theorem 21.7. ***(Taylor's Theorem.)*** *Suppose that f is continuous and has derivatives of all orders in a neighborhood of x_o, then f can be expressed as a power series*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-x_o)^n$$

with

$$a_n = \frac{f^{(n)}(x_o)}{n!} \quad .$$