CHAPTER 33

# Sample Final

**Math 2233**
SAMPLE FINAL EXAM

1. (15 pts) Solve the following initial value problem.

$$xy' + 3y = 5x \qquad , \qquad y(1) = 3 \quad .$$

- This is a first order linear equation with $p(x) = 3/x$ and $g(x) = 5$

$$\begin{aligned}
\mu &= \exp\left(\int p dx\right) = \exp\left(\int \frac{3}{x} dx\right) = \exp\left(3\ln(x)\right) = x^3 \\
y &= \frac{1}{\mu}\int \mu g dx + \frac{C}{\mu} = x^{-3}\int 5x^3 + Cx^{-3} = x^{-3}\frac{5}{4}x^4 = \frac{5}{4}x + Cx^{-3} \\
3 &= y(1) = \frac{5}{4} + C \qquad \Rightarrow \quad C = \frac{7}{4} \\
&\Rightarrow \qquad y = \frac{5}{4}x + \frac{7}{4x^3}
\end{aligned}$$

□

2. (15 pts) Find an implicit solution of the following initial value problem.

$$(y/x + 4x)\, dx + (\ln(x) - 3)\, dy = 0 \qquad , \qquad y(1) = 1$$

(Hint: the equation is exact.)

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$$\begin{aligned}
\Phi &= \int M\partial x + h_1(y) = \int (y/x + 4x)\,\partial x + h_1(y) = y\ln(x) + 2x^2 + h_1(y) \\
&= \int N\partial y + h_2(x) = \int (\ln(x) - 3)\,\partial y + h_2(x) = y\ln(x) - 3y + h_2(x) \\
&\Rightarrow \qquad h_1(y) = -3y \quad , \quad h_2(x) = 2x^2 \quad \Rightarrow \quad \Phi(x,y) = y\ln(x) + 2x^2 - 3y \\
&\Rightarrow \qquad y\ln(x) + 2x^2 - 3y = C \qquad \text{(implicit solution)}
\end{aligned}$$

□

3.

(a)(5 pts) Show that the following differential equation is not exact.

$$y + (2x - ye^y)\, y' = 0$$

- $$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y) = 1 \\ &\neq \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x - ye^y) = 2 \end{aligned}$$

□

(b) (10 pts) Find an integrating factor for the differential equation in (a). (Hint: look for an integrating factor that depends only on $y$.)

- $$\begin{aligned} F_2 &\equiv \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y}(2-1) = \frac{1}{y} \text{ does not depend on } x \\ &\Rightarrow \quad \mu(y) = \exp\left(\int F_2(y)dy\right) = \exp\left(\int \frac{1}{y}dy\right) = \exp(\ln(y)) = y \end{aligned}$$

□

4. (10 pts) Find an implicit solution of the following ODE.

$$\frac{dy}{dx} = \frac{y^2 + yx}{x^2} \quad .$$

(Hint: try the change of variables $z = y/x$.)

- $$\begin{aligned} y &= zx \quad \Rightarrow \quad y' = z'x + z \\ &\Rightarrow \quad z'x + z = y' = \frac{y^2 + yx}{x^2} = \frac{(zx)^2 + (zx)\,x}{x^2} = z^2 + z \\ &\Rightarrow \quad z'x + z = z^2 + z \quad \Rightarrow \quad z'x = z^2 \quad \Rightarrow \quad \frac{dz}{z^2} = \frac{dx}{x} \\ &\Rightarrow \quad \int \frac{dz}{z^2} - \int \frac{dx}{x} = C \quad \Rightarrow \quad -\frac{1}{z} - \ln(x) = C \\ &\Rightarrow \quad z = \frac{1}{-C - \ln(x)} \quad \Rightarrow \quad \frac{y}{x} = \frac{1}{-C - \ln(x)} \\ &\Rightarrow \quad y = \frac{-x}{C + \ln(x)} \end{aligned}$$

□

5. (15 pts) Given that $y_1(x) = x^{-1}$ is one solution of $x^2y'' + 3xy' + y = 0$, use Reduction of Order to determine the general solution and the solution satisfying $y(1) = 1$, $y'(1) = 0$.

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$$\begin{aligned} y_2 &= y_1 \int \frac{1}{(y_1)^2} \exp\left(-\int^x p ds\right) dx = x^{-1} \int x^2 \exp\left(\int^x -\frac{3}{s} ds\right) dx = x^{-1} \int x^2 \exp\left(-3\ln(x)\right) dx \\ &= x^{-1} \int x^2 x^{-3} dx = x^{-1} \int \frac{1}{x} dx = x^{-1} \ln(x) = x^{-1} \ln(x) \\ \Rightarrow &\quad y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 x^{-1} + c_2 x^{-1} \ln(x) \end{aligned}$$

□

6. (10 pts) Use the Method of Variation of Parameters to find the general solution of the following inhomogeneous differential equation.

$$y'' - 3y' + 2y = e^{3x} \quad .$$

- First we find two linear independent solutions of the corresponding homogeneous problem

$$\begin{aligned} y'' - 3y' + 2y &= 0 \quad \Rightarrow \quad y = e^{\lambda x} \quad \text{with } \lambda \text{ satisfying } 0 = \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) \\ &\Rightarrow \quad \lambda = 1, 2 \quad \Rightarrow \quad y_1 = e^x \quad , \quad y_2 = e^{2x} \end{aligned}$$

$$\begin{aligned} W[y_1, y_2] &= y_1 y_2' - y_1' y_2 = (e^x)\left(2e^{2x}\right) - (e^x)\left(2^{2x}\right) = (2-1)e^{3x} = e^{3x} \\ y_p &= -y_1 \int \frac{y_2 g}{W[y_1, y_2]} dx + y_2 \int \frac{y_1 g}{W[y_1, y_2]} dx \\ &= -e^x \int \frac{e^{2x} e^{3x}}{e^{3x}} dx + e^{2x} \int \frac{e^x e^{3x}}{3^{3x}} dx = -e^x \int e^{2x} dx + e^{2x} \int e^x dx \\ &= -e^x \left[\frac{1}{2} e^{2x}\right] + e^{2x} \left[e^x\right] = \frac{1}{2} e^{3x} \\ \Rightarrow &\quad y = y_p + c_1 y_1 + c_2 y_2 = \frac{1}{2} e^{3x} + c_1 e^x + c_2 e^{2x} \end{aligned}$$

□

7. (10 pts) What is the minimal radius of convergence of a power series solution of

$$(1+x^2)y'' + 2y' + xy = 0$$

about $x_o = 2$?

- The functions $p(x) = 2/\left(1+x^2\right)$ and $q\left(x\right) = x/\left(1+x^2\right)$ have denominators that vanish when $x = \pm i$. These singular points have coordinates $(0, \pm 1)$ in the complex plane and so are a distance

$$D = \sqrt{(2-0)^2 + (0 \mp 1)^2} = \sqrt{5}$$

from the point $(2, 0)$ (corresponding to the expansion point $x_0 = 2$ ). Thus,

$$R \geq \sqrt{5}$$

□

8.(15 pts) Find the recursion relations for a power series solution about $x_o = 1$ for the following differential equation.

$$xy'' - 2y = 0$$

- We set $y = \sum_{n=0}^{\infty} a_n \left(x-1\right)^n$ and demand

$$\begin{aligned}
0 &= xy'' - 2y = x\sum_{n=0}^{\infty} n\left(n-1\right)a_n\left(x-1\right)^{n-2} - 2\sum_{n=0}^{\infty} a_n\left(x-1\right)^n \\
&= \left[(x-1)+1\right]\sum_{n=0}^{\infty} n\left(n-1\right)a_n\left(x-1\right)^{n-2} + \sum_{n=0}^{\infty} -2a_n\left(x-1\right)^n \\
&= \sum_{n=0}^{\infty} n\left(n-1\right)a_n\left(x-1\right)^{n-1} + \sum_{n=0}^{\infty} n\left(n-1\right)a_n\left(x-1\right)^{n-2} + \sum_{n=0}^{\infty} -2a_n\left(x-1\right)^n \\
&= \sum_{n=-1}^{\infty} \left(n+1\right)\left(n\right)a_{n+1}\left(x-1\right)^n + \sum_{n=-2}^{\infty} \left(n+2\right)\left(n+1\right)a_{n+2}\left(x-1\right)^n + \sum_{n=0}^{\infty} -2a_n\left(x-1\right)^n \\
&= 0 + \sum_{n=0}^{\infty}\left(n+1\right)\left(n\right)a_{n+1}\left(x-1\right)^n + 0 + 0 + \sum_{n=0}^{\infty} \left(n+2\right)\left(n+1\right)a_{n+2}\left(x-1\right)^n + \sum_{n=0}^{\infty} -2a_n\left(x-1\right)^n \\
&= \sum_{n=0}^{\infty} \left[n\left(n+1\right)a_{n+1} + \left(n+2\right)\left(n+1\right)a_{n+2} - 2a_n\right]\left(x-1\right)^n \\
\Rightarrow &\quad a_{n+2} = \frac{2a_n - n\left(n+1\right)a_{n+1}}{\left(n+2\right)\left(n+1\right)} \qquad , \quad n = 0, 1, 2, 3, \ldots
\end{aligned}$$

□

9. (15 pts) Given that the recursion relations for $y'' - xy' + y = 0$ about $x_o = 0$ are

$$a_{n+2} = \frac{(n-1)a_n}{(n+2)(n+1)} \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

Write down the first 4 terms (i.e., to order $x^3$) for the power series solution satisfying $y(0) = 1$, $y'(0) = 2$.

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$$\begin{aligned}
y(0) &= 1 \quad \Rightarrow \quad a_0 = 1 \\
y'(0) &= 2 \quad \Rightarrow \quad a_1 = 2 \\
a_2 &= a_{0+2} = \frac{(0-1)\,a_0}{(0+2)(0+1)} = -\frac{a_0}{2} = -\frac{1}{2} \\
a_3 &= a_{1+3} = \frac{(1-1)a_1}{(1+2)(1+1)} = 0a_1 = 0 \\
y(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \\
&= 1 + 2x - \frac{1}{2}x^2 + 0 + \cdots
\end{aligned}$$

□

10. (15 pts) Consider the following linear differential equation $2xy'' - 2y = 0$. Assume a solution of the form $y = x^r \sum_{n=0}^{\infty} a_n x^n$ and determine the possible values of $r$.

- We set $y = \sum_{n=0}^{\infty} a_n x^{n+r}$ and demand

$$\begin{aligned}
0 &= 2xy'' - 2y = 2x \sum_{n=0}^{\infty} (n+r)(n+r-1)\, a_n x^{n+r-2} - 2 \sum_{n=0}^{\infty} a_n x^{n+r} \\
&= \sum_{n=0}^{\infty} 2\,(n+r)(n+r-1)\, a_n x^{n+r-1} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\
&= \sum_{n=-1}^{\infty} 2\,(n+r+1)(n+r)\, a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\
&= 2\,(-1+r+1)(-1+r)\, a_{-1+1} x^{-1+r} + \sum_{n=0}^{\infty} 2\,(n+r+1)(n+r)\, a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} -2a_n x^{n+r} \\
&= 2r\,(r-1)\, a_0 x^{r-1} + \sum_{n=0}^{\infty} \left[2\,(n+r+1)(n+r)\, a_{n+1} - 2a_n\right] x^{n+r}
\end{aligned}$$

So we need

$$0 = 2r\,(r-1) \quad \Rightarrow \quad r = 0, 1$$

□

11. (15 pts) Use the Laplace Transform technique to solve the following initial value problem.

$$y'' - 2y' - 3y = 0 \qquad , y(0) = 3 \quad , \quad y'(0) = 1$$

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$$\begin{aligned}
0 &= \mathcal{L}\left[y'' - 2y' - 3y\right] = \mathcal{L}\left[y''\right] - 2\mathcal{L}\left[y'\right] - 3\mathcal{L}\left[y\right] \\
&= \left(s^2\mathcal{L}\left[y\right] - sy(0) - y'(0)\right) - 2\left(s\mathcal{L}\left[y\right] - y(0)\right) - 3\mathcal{L}\left[y\right] \\
&= s^2\mathcal{L}\left[y\right] - 3s - 1 - 2s\mathcal{L}\left[y\right] + 6 - 3\mathcal{L}\left[y\right] \\
&= \left(s^2 - 2s - 3\right)\mathcal{L}\left[y\right] - 3s + 5 \\
&\Rightarrow \quad \mathcal{L}\left[y\right] = \frac{3s-5}{s^2-2s-3} = \frac{3s-5}{(s-3)(s+1)}
\end{aligned}$$

We use a Partial Fractions expansion to invert the Laplace transform of $(3s-5)/[(s-3)(s+1]$

$$\begin{aligned}
\frac{3s-5}{(s-3)(s+1)} &= \frac{A}{s-3} + \frac{B}{s+1} \quad \Rightarrow \quad 3s-5 = (s+1)A + (s-3)B \\
s &= -1 \quad \Rightarrow \quad -8 = 0 - 4B \quad \Rightarrow \quad B = 2 \\
s &= 3 \quad \Rightarrow \quad 4 = 4A + 0 \quad \Rightarrow \quad A = 1
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}\left[y\right] &= \frac{1}{s-3} + 2\frac{1}{s+1} = \mathcal{L}\left[e^{3x}\right] + 2\mathcal{L}\left[e^{-x}\right] = \mathcal{L}\left[e^{3x} + 2e^{-x}\right] \\
&\Rightarrow \quad y = e^{3x} + 2e^{-x}
\end{aligned}$$

□