Math 2233 - Lecture 2

Agenda:

- Classification of Differential Equations: Examples
- Graphical Interpretation of First Order ODEs
- Numerical Method for First Order ODEs

Classification of DEs: Examples

Classify the following differential equations: determine their order, if they are linear or non-linear, and if they are ordinary differential equations or partial differential equations.

(a) $y'' + \cos(y) = x$

- ODE , 2^{nd} order, nonlinear

(b) $\dfrac{\partial \phi}{\partial y} + \dfrac{\partial^2 \phi}{\partial x^2} = y^2$

- PDE , 2^{nd} order, linear

(c) $\dfrac{d^3 x}{dt^3} + x^2 \dfrac{dx}{dt} + x = 0$

- ODE , 3^{rd} order, non-linear

(d) $a(x)\, y' + b(x)\, y + c(x) = 0$

- ODE , 1^{st} order, linear

(e) $\frac{\partial \Phi}{\partial \xi} + \left(\frac{\partial \Phi}{\partial y}\right)^2 = \Phi$

- PDE , 1^{st} order , nonlinear

Standard Forms

It is easy to write down examples of first order, ordinary differential equations; such as

$$\begin{aligned} x\left(\frac{dy}{dx}\right) - e^{2x}y^2 &= \sin(x) \\ \log\left(y'\right) &= y\cos\left(x\right) \end{aligned}$$

However, these examples disguise somewhat the essence of a first order ODE.
Now, just like algebraic equations, it is possible to manipulate a differential equations in such a way that their solutions don't change.
(As a general rule, anytime you do the same mathematical operation to both sides of an equation, you get an equivalent equation; i.e., an equation with the same solutions)

Standard Forms, Cont'd

As we now begin to discuss how to **solve** differential equations, it convenient to assume that we have already carried out algebraic manipulations that have both

- simplied as much as possible, the essential form of a differential equation, and
- provided a common starting point for solving the multitude of equivalent equations.

In the case of 1st order ODEs, this most basic, common, **standard form** is

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

Our study of first order ordinary differential equations, therefore, begins with looking for solutions of (1)

Example: Converting a 1st Order ODE to Standard Form

Consider

$$\left(y'\right)^2 \sin(x) = \cos(xy)$$

If we "solve this equation for y'" we get

$$y' = \pm\sqrt{\frac{\cos(xy)}{\sin(x)}}$$

Note that the latter equation is in the desired "standard form", with

$$F(x, y) = \pm\sqrt{\frac{\cos(xy)}{\sin(x)}}$$

As the course progresses, we'll solve large families of differential equations, one-by-one, by first specifying a common standard form, and then working out the details of the solution for that fixed standard form.

Graphical Interpretation of First Order ODEs

Our goal now is to find functions $y(x)$ satisfying a differential equation of the form

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

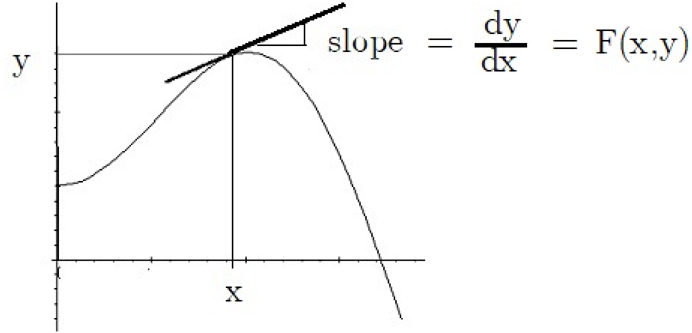
However, even in this simple form, the case of 1st Order ODEs is too difficult to be solved completely.
Yet, (1) is at least specific enough for us to extract some qualitative information about its solutions.
I'll next show how equation (1) can be used a figure out what the graph of any of its solutions will look like.
In fact, the graphical method I'm about to present can be regarded as method for constructing the graph of an solution (effectively, a "visualization of the solution" rather than an explicit function)

Graphical Interpretation of First Order ODEs, Cont'd

Let us consider what the differential equation

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

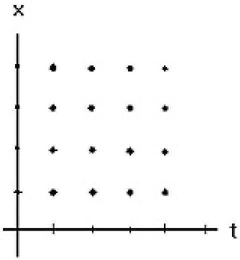
says about the graph of a solution function $y(x)$.
From Calculus I, we know the LHS (left hand side) of (1) is interpretable as the **slope** of the tangent line to the graph of $y(x)$ at the point $(x, y(x))$ along its graph.



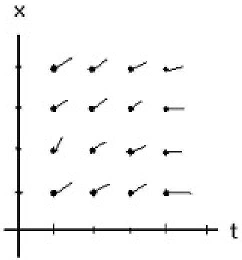
Graphical Interpretation of First Order ODEs, Cont'd

Put another way, if the graph of a solution passes through the point (x, y) then it must pass through that point with slope $F(x, y)$.

Let's remove, for the moment, the graph of the solution from the picture, just so we can set up a more complete picture of the solution. First, we form a nice rectangular grid of "sample" points (x_i, y_j) in the xy-plane.



Next, we calculate the value of the function $F(x, y)$ (the function on the right hand side of the ODE) at each grid point (x_i, y_j).
By virtue of the ODE it satisfies, if a solution passes through the grid point (x_i, y_j) it must do so with slope equal to the number $F(x_i, y_j)$.
We indicate this on our figure by drawing a short arrow with the proper slope at each of our grid points.



We call this figure with its grid of points and attached arrows, a **direction field plot** for the differential equation.

I admit the preceding figure looks a bit crude and uninformative, but that's because only 16 sample points were used.
With the aid of a computer, one can handle hundreds of sample points, and one can produce direction field plots like

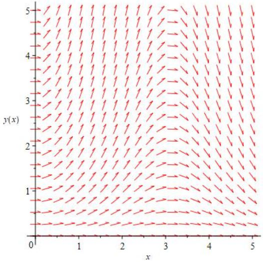


Figure: The Direction Field Plot for $\frac{dy}{dx} = y\sin(x)$

With such a figure in hand, we can now readily sketch out solutions to the differential equation.

One simply has to put one's pencil down on the graph and draw a curve that always follows the direction of the arrows closest to it.

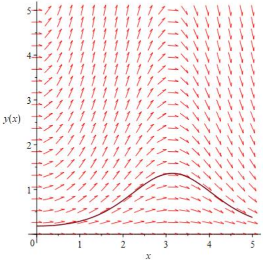


Figure: A Sketch of a Solution to $\frac{dy}{dx} = y\sin(x)$

The resulting curve then provides a good sketch of the graph of an actual solution to the differential equation.
In fact, by simply choosing different starting points, we can readily use the above direction field plot to sketch out multiple solutions of the differential equation.

So while we do not yet know the explicit functional form of a solution $y(x)$ that satisfies the differential equations $\frac{dy}{dx} = y \sin(x)$, we can at least figure out what the (graphs of) solutions must look like.

This graphical methodology also tells us something vitally important to our understanding of first order ODEs.

Given a 1st Order ODE (i.e., given a function $F(x, y) fortheR.H.S.of (1)$, once we choose a starting point, or **initial condition**, for a solution, the its graph is completely determined by the direction field plot, which in turn is prescribed by the function $F(x, y)$.

Existence and Uniqueness Theorem for 1st Order ODEs

In fact, we have

Theorem

So long as the function $F(x, y)$ is well-defined and continuous with respect to its variables, there will be one, and only one, solution to

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

satisfying a given initial condition

$$y(x_0) = y_0 \tag{2}$$

This theorem, by the way, closes the theoretical gap I mentioned in the first lecture. With this theorem in hand, anytime we find can find a function form for $y(x)$ that

- satisfies the differential equation (1),
- satisfies all possible initial conditions (2)

we will have found **all** of the solutions of the differential equation.

A Numerical Method for solving $\frac{dy}{dx} = F(x, y)$

The graphical interpretation of first order ODEs just developed can also be leveraged to determine a table of approximate values for a solution $y(x)$.

Let us again interpret the differential equation

$$\frac{dy}{dx} = F(x, y)$$

as saying that if a solution passes thru the point (x_0, y_0) in the xy-plane, then it does so with slope $F(x_0, y_0)$.

Now the tangent line to the graph of the solution at (x_0, y_0) is interpretable as the best straight line fit to the actual solution near the point (x_0, y_0).

If we follow this straight line approximation for $y(x)$ out a bit by increasing x_0 to $x_1 = x_0 + \Delta x$, then along that line, y will change by

$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x = (slope)\,\Delta x \approx \frac{dy}{dx}\Delta x = F(x_0, y_0)\,\Delta x$$

Thus, if we set

$$\begin{aligned} x_1 &= x_0 + \Delta x \\ y_1 &= y_0 + \Delta y = y_0 + F(x_0, y_0)\, \Delta x \end{aligned}$$

then the point (x_1, y_1) will also live on this best straight line fit to the actual solution and so should be a good approximation to a point on the actual solution. In other words, $y(x_1) \approx y_1$. We can now repeat the process to find more approximate values for $y(x)$,

- regard (x_1, y_1) as a point on the graph of a solution
- compute the slope of the solution at (x_1, y_1) as $F(x_1, y_1)$,
- find a new line that best approximates the actual solution $y(x)$ near the point (x_1, y_1), and
- then use that line to find another approximate point on the solution graph

$$\begin{aligned} x_2 &= x_1 + \Delta x \\ y_2 &= y_1 + F(x_1, y_1)\, \Delta x \end{aligned}$$

Repeating this process over and over, we can readily generate a table of approximate values for the solution of the stated differerential equation and initial condition.

Example

Consider the following initial value problem.

$$\begin{aligned} y' &= x^2 y \\ y(1) &= 2 \end{aligned}$$

Estimate $y(1.3)$ using step sizes of $\Delta x = 0.1$.

The initial condition gives an initial value, 1, for x and an initial value, 2, for y. Thus we set,

$$\begin{aligned} x_0 &= 1 \\ y_0 &= 2 \end{aligned}$$

Next, we increase x to

$$x_1 = x_0 + \Delta x = 1 + 0.1 = 1.1$$

and try the follow the best straight line fit to the actual solution through $(1, 2)$. We'll then arrive at

$$y_1 = y_0 + F(x_0, y_0)\,\Delta x = 2 + x^2 y\Big|_{\substack{x=1\\y=2}}\Delta x = 2 + (1)^2\,(2)\,(0.1) = 2.2$$

Thus,

$$y(1.1) \approx 2.2$$

Now increase x again to $x_2 = x_1 + \Delta x = 1.2$. The corresponding value of y will be

$$y_2 = y_1 + F(x_1, y_1)\,\Delta x = 2.2 + (1.1)^2\,(2.2)\,(0.1) = 2.4662$$

So

$$y(1.2) \approx 2.4662$$

Repeating this process one more time, we get

$$\begin{aligned} x_3 &= x_2 + \Delta x = 1.3 \\ y_3 &= y_2 + F(x_2, y_2)\,\Delta x = 2.4662 + \ (1.2)^2 (2.4662)(0.1) = 2.\,821\,3 \end{aligned}$$

and we can conclude

$$y(1.3) \approx 2.8213$$

Euler's Method: Summary

Given

$$\begin{aligned} y' &= F(x, y) \\ y(x_0) &= y_0 \end{aligned}$$

Choose a (small) Δx and set

$$\begin{aligned} x_1 &= x_0 + \Delta x \\ y_1 &= y_0 + F(x_0, y_0)\,\Delta x \end{aligned}$$

and then compute successive pairs (x_i, y_i) via

$$\begin{aligned} x_i &= x_{i-1} + \Delta x \\ y_i &= y_{i-1} + F(x_{i-1}, y_{i-1})\,\Delta x \end{aligned}$$

Each time you do this you get a new approximate point on the solution of the initial value problem.
This will allow you to build up a table of (approximate) values for the actual solution.