Math 2233 - Lecture 2

Agenda:

- Classification of Differential Equations: Examples
- Graphical Interpretation of First Order ODEs
- Numerical Method for First Order ODEs

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As the course progresses, we'll solve large families of differential equations, one-by-one, by first specifying a common standard form, and then working out the details of the solution for that fixed standard form.

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In fact, the graphical method I'm about to present can be regarded as method for constructing the graph of an solution (effectively, a "visualization of the solution" rather than an explicit function)

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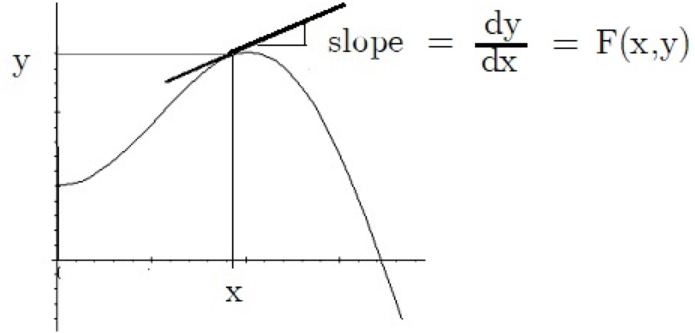

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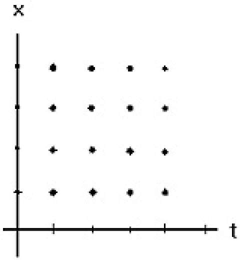

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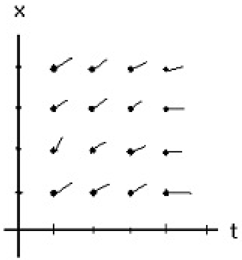

We call this figure with its grid of points and attached arrows, a **direction field plot** for the differential equation.

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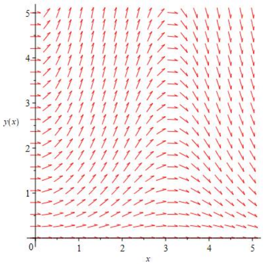

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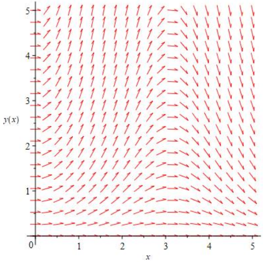

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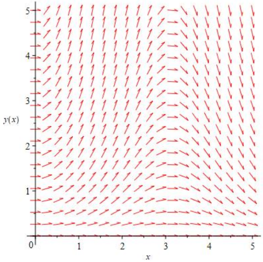

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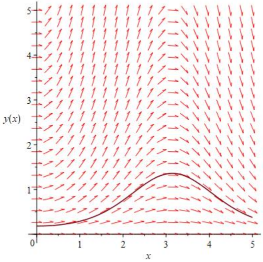

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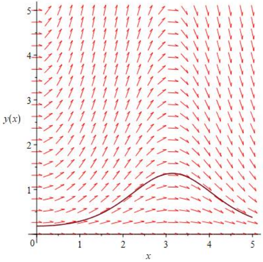

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The resulting curve then provides a good sketch of the graph of an actual solution to the differential equation.

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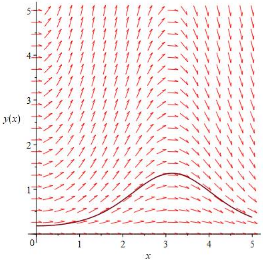

Figure: A Sketch of a Solution to $\frac{dy}{dx} = y\sin(x)$

The resulting curve then provides a good sketch of the graph of an actual solution to the differential equation.
In fact, by simply choosing different starting points, we can readily use the above direction field plot to sketch out multiple solutions of the differential equation.

So while we do not yet know the explicit functional form of a solution $y(x)$ that satisfies the differential equations $\frac{dy}{dx} = y \sin(x)$, we can at least figure out what the (graphs of) solutions must look like.

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Given a 1st Order ODE (i.e., given a function $F(x, y) for the R.H.S. of (1)$, once we choose a starting point, or **initial condition**, for a solution, the its graph is completely determined by the direction field plot, which in turn is prescribed by the function $F(x, y)$.

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So long as the function $F(x, y)$ is well-defined and continuous with respect to its variables, there will be one, and only one, solution to

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This theorem, by the way, closes the theoretical gap I mentioned in the first lecture. With this theorem in hand, anytime we find can find a function form for $y(x)$ that

- satisfies the differential equation (1),
- satisfies all possible initial conditions (2)

we will have found **all** of the solutions of the differential equation.

A Numerical Method for solving $\frac{dy}{dx} = F(x, y)$

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Let us again interpret the differential equation

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as saying that if a solution passes thru the point (x_0, y_0) in the xy-plane, then it does so with slope $F(x_0, y_0)$.

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$$\Delta y = \frac{\Delta y}{\Delta x}\Delta x = (slope)\,\Delta x \approx \frac{dy}{dx}\Delta x = F(x_0, y_0)\,\Delta x$$

Thus, if we set

$$
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Consider the following initial value problem.

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Repeating this process one more time, we get

$$\begin{aligned} x_3 &= x_2 + \Delta x = 1.3 \\ y_3 &= y_2 + F(x_2, y_2)\,\Delta x = 2.4662 + (1.2)^2 (2.4662)(0.1) = 2.8213 \end{aligned}$$

and we can conclude

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This will allow you to build up a table of (approximate) values for the actual solution.