Agenda:

- 1. MyLab Math HW1 demo
- 2. First Order ODEs : Exact Solutions
- 3. The Fundamental Theorem of Calculus

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## First Order ODEs : Exact Solutions

- We now turn our attention to the problem of finding explicit formulas for functions that satisfy a first order ODE. (as opposed to finding the graphs of a solution or a table of approximate values for a solution).
- Recall that our standard form for a 1st order ODE is

$$\frac{dy}{dx} = F(x, y) \tag{1}$$

- Even this simple form is too difficult to solve for most functions F(x, y).
- So instead, we will begin by looking some special cases of (1)
- Then as our experience with the special cases grows, we be able to handle more complicated cases of (1).

Case 1: F(x,y) = f(x): Applying the Fundamental Theorem of Calculus

Let us now specialize to the case where the function F(x, y) on the R.H.S. (right hand side) of (1) actually only depends on the underlying variable x, and not on the unknown function y.

$$\frac{dy}{dx} = f(x) \tag{2}$$

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It turns out that a solution to (2) can be readily found by simply integrating both sides of this equation with respect to x. Indeed, suppose we multiply both sides of (2) by dx and then integrate both sides with respect to x.

$$\int \frac{dy}{dx} dx = \int f(x) dx$$

Now according to the Fundamental Theorem of Calculus, the left hand side of the preceding equation is equal to

$$\int \frac{dy}{dx} dx = y(x)$$

Thus,

$$y(x) = \int \frac{dy}{dx} dx = \int f(x) dx$$

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is a solution to the differential equation.

## Example

$$\frac{dy}{dx} = x\cos\left(x\right)$$

We should have

$$y(x) = \int x \cos(x) \, dx$$

Using integration-by-parts

$$\int u \, dv = uv - \int v \, du$$

with

$$u = x , \quad dv = \cos(x) dx$$
  
$$du = dx , \quad v = \int dv = \int \cos(x) dx = \sin(x)$$

we find

$$\int x \cos(x) \, dx = x \sin x - \int \sin(x) \, dx = x \sin(x) - \cos(x)$$

## Example, Cont'd

So

$$y(x) = x\sin(x) - \cos(x)$$

is our solution. This is easily verified:

$$\frac{d}{dx}(x\sin(x) - \cos(x)) = (\sin(x) + x\cos(x)) - \sin(x) = x\cos(x)$$

and so we indeed have found a solution.

But last week I told you that if a differential equation has a solution, it will actually have infinitely many solutions. So where are the other solutions?

Well, we won't have to look too hard for them. All we really need is more nuanced version of the Fundamental Theorem of Calculus.

### Digression: Definite Integrals

Suppose f(x) is a continuous function on the closed interval [a, b]. If f(x) is a positive function, then the **definite integral**  $\int_a^b f(x) dx$  calculates the area under the graph of f(x) between x = a and x = b. But more generally,  $\int_a^b f(x) dx$  calculates the difference between the areas  $A_+$  and  $A_-$  that lie between the graph of f(x) and the x-axis, and between the lines x = a and x = b.



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## Definite Integrals, Cont'd

#### One has

$$\int_{a}^{b} f(x) dx \equiv \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x \qquad , \qquad \Delta x = \frac{b-a}{N}$$

and then

$$\int_a^b f(x) dx = A_+ - A_-$$

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Note the result here is just a number.

## Indefinite Integrals (Anti-Derivatives)

There is another type of integral, the **indefinite integral**, that produces a function of x rather than just a number. The notation for indefinite integrals are is like that of definite integrals except there are no endpoints of integration

$$\int f(x) \, dx \tag{3}$$

and the function  $F(x) = \int f(x) dx$  produced by an indefinite integral is, by definition, an **anti-derivative** of the integrand f(x); i.e., F satisfies

$$\frac{dF}{dx} = f(x)$$

 $N.B.,\ So,\ definite\ integrals\ and\ indefinite\ integrals,\ while\ having\ a\ similar\ notation,\ are\ quite\ different\ conceptually.$ 

• Definite integrals  $\int_a^b f(x) dx$  computes a difference of areas by taking a limit

• Indefinite integrals  $\int f(x) dx$  yields an anti-derivative of the function f.

The indefinite integral notation (3) for anti-derivatives, however, can be a bit misleading; because it suggests there is just one anti-derivative of a function f(x). But if F(x) is an anti-derivative of x and C is a constant

$$\frac{d}{dx}\left(F\left(x\right)+C\right)=f\left(x\right)+0=f\left(x\right)$$

so is any function of the form F(x) + C will also be an anti-derivative of f(x). (It is because of **this** ambiguity, that we call the integral (3) an **indefinite** integral).

## Digression: Definite and Indefinite Integrals, Cont,d

There is yet another kind of integral that sort of interpolates between the definite integrals and indefinite integrals. By regarding the upper endpoint of integration in  $\int_a^b f(x) dx$  as a variables, we can regard the corresponding family of definite integrals as a function of x

$$F_{a}\left(x\right) = \int_{a}^{x} f\left(x'\right) dx'$$

(the "dummy integration variable" x in (3) has been renamed as x' so its role can be distinguished from the x used as the endpoint of integration). This integral still computes the area under the graph of f(x') between x' = a and x' = x; but we now allow x to vary.

With this detailed notation/nomenclature established, I can now state the Fundamental Theorem of Calculus as it appears in Calculus I.

#### Theorem

Suppose f(x) is a continuous function on the closed interval [a, b]and let  $F_a(x)$  be defined as above. Then

(i) 
$$\frac{dF_a}{dx}(x) = f(x)$$
  
Thus,  $F_a(x)$  is an anti-derivative of  $f(x)$ .  
(ii) Suppose  $\widetilde{F}$  is any anti-derivative of  $f(x)$ , then  
 $\int_a^b f(x) = \widetilde{F}(b) - \widetilde{F}(a)$ 

(i) Tells us how a definite integrals lead to indefinite integrals (anti-derivatives)

(ii) Tells us how we can use indefinite integrals (anti-derivatives) to compute definite integrals.

# The Fundamental Theorem of Calculus and 1st Order ODEs

Let me now state the Fundamental Theorem of Calculus in a way that more suitable for a differential equations course.

#### Theorem

The general solution of

$$\frac{dy}{dx} = f(x) \tag{4}$$

is

$$y\left(x\right)=\int f\left(x\right)dx+C$$

where C is an arbitrary constant.

Note that (5) says that y(x) must be an anti-derivative of f(x), and that RHS of (5) is just a way of formulating the most general anti-derivative of f(x).

### Example

Consider

$$\frac{dy}{dx} = x^2 + 2x$$

(a) Find the general solution of this first order ODE. (b) Find the unique solution satisfying y(0) = 1

According to the preceding theorem, the general solution will be

$$y(x) = \int (x^2 + 2x) dx + C = \frac{1}{3}x^3 + x^2 + C$$

This answers (a). This general solution constitutes the functional form of every solution of the differential equation. In particular, a solution satisfying y(0) = 1, should have this functional form - but it may also require a special value for *C*.

## Example, Cont'd

So, let's see what's needed for the constant C. If we start with the form of the general solution

$$y(x) = \frac{1}{3}x^3 + x^2 + C$$

and impose the condition y(0) = 1,

$$1 = y(0) = \frac{1}{3}(0)^{3} + (0)^{2} + C = C$$
  
$$\Rightarrow \quad C = 1$$

Thus,

$$y(x) = \frac{1}{3}x^3 + x^2 + 1$$

will be the unique function satisfying both the differential equation and the initial condition.

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Summary: Solving  $\frac{dy}{dx} = f(x)$ 

 (i) The general solution is obtained by integrating f(x) with respect to x and then adding "by hand" an arbitrary constant C

$$y(x) = \int f(x)dx + C$$
 (5)

(ii) To find the solution satisfying a particular initial condition

$$y(x_0) = y_0 \tag{6}$$

one inserts the general solution (5) into (6) and solves for the appropriate value for constant C. The desired solution is then the function obtained by replacing the unspecified constant C in the general solution by that particular number that guarantees the validity of (6).

### Separable First Order ODEs

Let's now move on to another special case of first order ODEs. We'll start with the standard form of a first order ODE

$$\frac{dy}{dx} = F(x, y)$$

and immediately specialize to the case when the function F on the right hand side is the quotient of a function M of x alone and a function N of y alone.

$$F(x,y) = -\frac{M(x)}{N(y)}$$
(8)

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Such equations (as well as equations that can be transformed into this form) are called **Separable Equations**.

## Separable First Order ODEs, Cont'd

Another way of writing a separable equation is

$$M(x) + N(y)\frac{dy}{dx} = 0$$
(9)

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This latter form shows you why these equations are called "separable": because the *y*-dependent terms of the equation can be completely separated from the *x*-dependent terms.

Below I'll give a quick method for getting to the solution of a separable equation

It's only drawback is that is doesn't really hold up to mathematical scrutiny step-by-step.

Nevertheless, it still works.

Once I've demonstrated for you the quick and dirty method, I'll circle back and provide a more palatable mathematical explanation of the method.

# Solving Separable Equations : The Mneumonic Method, Cont'd

Let me start with a separable equation in a form equivalent to (9).

$$N(y)\frac{dy}{dx} = -M(x) \tag{9}$$

Multiplying both sides by dx we get

$$N(y) dy = -M(x) dx$$

If we now integrate both sides with respect to their respective variables we get

$$\int N(y) \, dy = -\int M(x) \, dx \tag{10}$$

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Notice that we have gotten rid of the derivative of y.

# Solving Separable Equations : The Mneumonic Method, Cont'd

What remains are two computable functions

$$H_1(x) \equiv \int M(x) dx$$
  
$$H_2(y) \equiv \int N(y) dy$$

which must be related by

$$H_2(y) = -H_1(x)$$
 (11)

But now (11) places an algebraic condition on the variables x and y. Effectively, we have replaced the original differential equation by an equivalent algebraic equation. If we now solve (algebraically) equation (10) for y, we'll end up with a solution of the differential equation.

# Example: Solving a Separable Equation via the Mneumonic Method

Consider

$$y\frac{dy}{dx} = \cos(x)$$

This is in the form (9) of a separable equation with

$$M(x) = cos(x)$$
 ,  $N(y) = y$ .

Multiplying both sides by dx and integrating we get

$$\frac{1}{2}y^2 = \int y \, dy = \int \cos(x) \, dx = \sin(x)$$

Solving the extreme sides of this last equation for y, we get

$$y=\pm\sqrt{2\sin(x)}$$

and indeed both

$$y(x) = \sqrt{2\sin(x)}$$
  
$$y(x) = -\sqrt{2\sin(x)}$$

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satisfy the differential equation.

## Critique of the Mneumonic Method

Our Mneumonic Method rapidly produces solutions to a separable equation. However, a few of the steps are a bit questionable, mathematically speaking.

- What does it mean to multiply both sides by dx? dx is notational device not a number
- How can one integrate one side of an equation with respect to y and the other side with respect to x and expect to maintain consistency?
- Where are the other solutions? (Introducing an arbitrary constant C at the end does not produce new solutions)

In the next lecture I'll show you the details of why this method works and how we find *all* of the solutions.