Math 2233 - Lecture 3

Agenda:

1. MyLab Math HW1 demo
2. First Order ODEs : Exact Solutions
3. The Fundamental Theorem of Calculus

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- So instead, we will begin by looking some special cases of (1)
- Then as our experience with the special cases grows, we be able to handle more complicated cases of (1).

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It turns out that a solution to (2) can be readily found by simply integrating both sides of this equation with respect to x. Indeed, suppose we multiply both sides of (2) by dx and then integrate both sides with respect to x.

$$\int \frac{dy}{dx} dx = \int f(x) dx$$

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Thus,

$$y(x) = \int \frac{dy}{dx} dx = \int f(x)\, dx$$

is a solution to the differential equation.

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we find

$$\int x\cos(x)\,dx = x\sin x - \int \sin(x)\,dx = x\sin(x) - \cos(x)$$

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This is easily verified:

$$\frac{d}{dx}\left(x\sin(x) - \cos(x)\right) = \left(\sin(x) + x\cos(x)\right) - \sin(x) = x\cos(x)$$

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Well, we won't have to look too hard for them. All we really need is more nuanced version of the Fundamental Theorem of Calculus.

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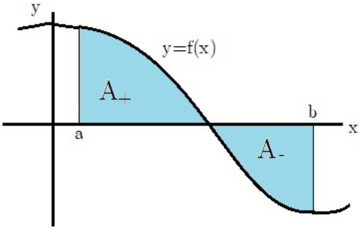

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Note the result here is just a number.

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so is any function of the form $F(x) + C$ will also be an anti-derivative of $f(x)$.

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so is any function of the form $F(x)+C$ will also be an anti-derivative of $f(x)$. (It is because of **this** ambiguity, that we call the integral (3) an **indefinite** integral).

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(the "dummy integration variable" x in (3) has been renamed as x' so its role can be distinguished from the x used as the endpoint of integration). This integral still computes the area under the graph of $f\left(x'\right)$ between $x' = a$ and $x' = x$; but we now allow x to vary.

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(ii) Tells us how we can use indefinite integrals (anti-derivatives) to compute definite integrals.

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Note that (5) says that y(x) must be an anti-derivative of $f(x)$, and that RHS of (5) is just a way of formulating the most general anti-derivative of $f(x)$.

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Thus,

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will be the unique function satisfying both the differential equation and the initial condition.

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one inserts the general solution (5) into (6) and solves for the appropriate value for constant C. The desired solution is then the function obtained by replacing the unspecified constant C in the general solution by that particular number that guarantees the validity of (6).

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and immediately specialize to the case when the function F on the right hand side is the quotient of a function M of x alone and a function N of y alone.

$$F(x, y) = -\frac{M(x)}{N(y)} \tag{8}$$

Separable First Order ODEs

Let's now move on to another special case of first order ODEs.

We'll start with the standard form of a first order ODE

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Such equations (as well as equations that can be transformed into this form) are called **Separable Equations**.

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This latter form shows you why these equations are called "separable" : because the y-dependent terms of the equation can be completely separated from the x-dependent terms.

Solving Separable Equations : The Mneumonic Method

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Nevertheless, it still works.

Once I've demonstrated for you the quick and dirty method, I'll circle back and provide a more palatable mathematical explanation of the method.

Solving Separable Equations : The Mneumonic Method, Cont'd

Let me start with a separable equation in a form equivalent to (9).

$$N(y)\frac{dy}{dx} = -M(x) \tag{9}$$

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Notice that we have gotten rid of the derivative of y.

Solving Separable Equations : The Mneumonic Method, Cont'd

What remains are two computable functions

$$H_1(x) \equiv \int M(x)\,dx$$
$$H_2(y) \equiv \int N(y)\,dy$$

which must be related by

$$H_2(y) = -H_1(x) \tag{11}$$

But now (11) places an algebraic condition on the variables x and y. Effectively, we have replaced the original differential equation by an equivalent algebraic equation. If we now solve (algebraically) equation (10) for y, we'll end up with a solution of the differential equation.

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and indeed both

$$\begin{aligned} y(x) &= \sqrt{2\sin(x)} \\ y(x) &= -\sqrt{2\sin(x)} \end{aligned}$$

satisfy the differential equation.

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In the next lecture I'll show you the details of why this method works and how we find *all* of the solutions.