Math 2233 - Lecture 4

Agenda:

1. Separable Equations: Theory
2. Examples of Solving Separable Equation
3. 1st Order Linear ODEs, Intro

Separable First Order ODEs

At the end of the last lecture we considered the case of 1st Order Separable ODEs:
i.e., differential equations that can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{1}$$

I also gave a quick and dirty method for writing down some exact solutions; the so-called Mneumonic Method, whereby, one converts the original separable differential equation (1) to an equivalent algebraic equation.
The equivalent algebraic equation is then used to solve for y; and the resulting formula for y will yield a function of x that solves the original differential equation.

The Mneumonic Method

1. Rewrite the differential equation as

$$N(y)\frac{dy}{dx} = -M(x)$$

2. Multiply both sides by dx to get

$$N(y)\,dy = -M(x)\,dx$$

3. Now integrate both sides

$$\int N(y)\,dy = \int -M(x)\,dx \qquad (3)$$

4. Let $H_1(x) \equiv \int M(x)\,dx$ and let $H_2(y) \equiv \int N(y)\,dy$, then (3) is equivalent to

$$H_1(x) + H_2(y) = 0 \qquad (2)$$

5. An exact solution of the original separable equation is found by solving (2) for y.

Critique of the Mneumonic Method

Our Mneumonic Method rapidly produces solutions to a separable equation.
However, a few of the steps are a bit questionable, mathematically speaking.

- What does it mean to multiply both sides by dx? dx is notational device not a number
- How can one integrate one side of an equation with respect to y and the other side with respect to x and expect to maintain consistency?
- Where are the other solutions?
  (Introducing an arbitrary constant C at the end does not produce new solutions)

Today, I'll show you the details of why this method works and how we find *all* of the solutions.

Separable Equations Understood Properly

The essence of the Mneumonic Method is that it purports to solve a differential equation by solving instead an equivalent algebraic equation.

This raises another question: How can a differential equation be related to an algebraic equation?

Digression: Implicit Differentiation

Suppose you have an algebraic equation, like

$$x^2 + y^2 = 1$$

If we solve for y, we end up prescribing the variable y as a certain function of x

$$y = \pm\sqrt{1 - x^2}$$

Let's make this interpretation of y explicit in the original equation by writing

$$x^2 + (y(x))^2 = 1$$

Now, differentiate both sides of this equation with respect to x (this is a valid mathematical operation, since we are doing **exactly the same thing** to both sides of an equality)

$$\frac{d}{dx}\left(x^2 + (y(x))^2\right) \quad = \quad \frac{d}{dx}(1)$$

$$\Longrightarrow \quad 2x + 2 * y(x)\frac{dy}{dx} = 0$$

Digression: Implicit Differentiation, Cont'd

Note how the Chain Rule

$$\frac{d}{dx}\left(h\left(y(x)\right)\right) = \frac{dh}{dy}\frac{dy}{dx}$$

was employed to compute $\frac{d}{dx}\left(y(x)\right)^2$. as $2y\frac{dy}{dx}$)

We have thus **derived** a differential equation from the original algebraic equation.

It follows that solutions of the differential equation will also be solutions the original algebraic equation.

Constructing Separable ODEs by Implicit Differentiation

This process by which we differentiate an algebraic equation to yield a differential equation is called **implicit differentiation**.

Suppose we now start with an algebraic equation of the form

$$H_1(x) + H_2(y) = C \qquad (*)$$

where C is a constant.

If we carry out implicit differentiation of this equation, we get

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

Note that the resulting differential equation is of the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

with

$$M(x) = H_1'(x) \qquad , \qquad N(y) = H_2'(y)$$

Thus, implicit differentiation of algebraic equation of the form (*) produces a separable ODE with the same solution set.

Deriving an Algebraic Equation from a Separable ODE

However, we want to reverse this process: We want to derive an equivalent algebraic equation from a separable ODE.
So let's now suppose that a separable differential equation

$$M(x) + N(y)\frac{dy}{dx} = 0$$

was derived from an algebraic equation of the form

$$H_1(x) + H_2(y) = C \qquad (**)$$

Then, as we showed above, we must have

$$\begin{aligned} M(x) &= H_1'(x) \\ N(y) &= H_2'(y) \end{aligned}$$

Let us now think of these two equations as differential equations for $H_1(x)$ and $H_2(y)$. Both of these equations are readily solved using the Fundamental Theorem of Calculus (as we discussed at the beginning of this lecture)

Deriving an Algebraic Equation from a Separable ODE, Cont'd

$$
\begin{aligned}
M(x) &= H_1'(x) \quad \Longrightarrow \quad H_1(x) = \int M(x)\, dx + c_1 \\
N(y) &= H_2'(y) \quad \Longrightarrow \quad H_2(y) = \int N(y)\, dy + c_2
\end{aligned}
$$

Substituting these results into the original algebraic equation we get

$$\int M(x)\, dx + c_1 \; + \; \int N(y)\, dy + c_2 = C$$

Consolidating the three additive arbitrary constants into a single arbitrary constant, by effectively absorbing the values of c_1 and c_2 into C, we end up with

$$\int M(x)\,dx \;+\; \int N(y)\,dy = C$$

When C equals 0 we get the same algebraic equation that was produced by our (mathematically improper) Mneumonic Method. Moreover, we now see how to get more solutions from our Mneumonic method; for, in the present derivation, C is an arbitrary constant, not necessarily equal to 0.

Summary: Solving Separable Equations

Let

$$M(x) + N(y)\frac{dy}{dx} = 0$$

be a Separable ODE.

The following procedure solves this ODE:

1. Compute

$$\begin{aligned} H_1(x) &= \int M(x)\, dx \\ H_2(y) &= \int N(y)\, dy \end{aligned}$$

2. Set up the equivalent algebraic equation

$$H_1(x) + H_2(y) = C$$

3. Solve the algebraic equation for y as a function of x and C.

Example 1

Show that the following equation is separable and then solve it.

$$y\cos(x) + sin(x)\frac{dy}{dx} = 0$$

This equation is not yet in explicit separable form.
We need to separate the x-dependency from the y-dependency.
This we do by multiplying both sides by $\frac{1}{y\sin(x)}$.

$$\frac{\cos(x)}{\sin(x)} + \frac{1}{y}\frac{dy}{dx} = 0 \qquad (4)$$

(4) is of Separable Form, with

$$M(x) = \frac{\cos(x)}{\sin(x)} \qquad , \qquad N(y) = \frac{1}{y}$$

Example 1, Cont'd

We thus need to solve (4) as a Separable Equation.
We have

$$\begin{aligned} H_1(x) &\equiv \int M(x)\,dx = \int \frac{\cos(x)}{\sin(x)}dx = \ln(\sin(x)) \\ H_2(y) &\equiv \int N(y)\,dy = \int \frac{1}{y}dy = \ln|y| \end{aligned}$$

So (4) is equivalent to the following algebraic algebraic equation

$$H_1(x) + H_2(y) = C \quad \Rightarrow \quad \ln(\sin(x)) + \ln(y) = C$$

Solving this last equation of y, we get

$$\ln|y| = C - \ln(\sin(x))$$

or, after exponentiating both sides,

$$\exp(\ln|y|) = \exp(C - \ln(\sin(x))) \tag{5}$$

Example 1, Cont'd

Next, we use some identities for exponential and logarithmic functions

$$
\begin{aligned}
\exp(C) &\equiv e^{C} \\
\exp(\ln|x|) &= x \\
\exp(A+B) &= \exp(A)\exp(B) \\
\exp(\lambda B) &= \exp(B)^{\lambda}
\end{aligned}
$$

to simplify both sides of (5) We find

$$LHS = \exp(\ln|y|) = y$$

and

$$
\begin{aligned}
RHS &= \exp(C - \ln(\sin(x))) \\
&= \exp(C)\exp(-\ln(\sin(x))) \\
&= \exp(C)(\exp(\ln(\sin(x))))^{-1} \\
&= e^{C}(\sin(x))^{-1} \\
&= \frac{e^{C}}{\sin(x)}
\end{aligned}
$$

Example 1, Cont'd

Equating the left and right hand sides, we get

$$y(x) = \frac{e^C}{\sin(x)}$$

This is the functional form of the general solution to the differential equation.

Example 2

Find the solution of

$$\frac{dy}{dx} = y^2/x \tag{6}$$

satisfying the initial condition

$$y(1) = 2 \tag{7}$$

Recasting this (6) in explicitly separable form, we have

$$-\frac{1}{x} + \frac{1}{y^2}\frac{dy}{dx} = 0$$

Thus,

$$\begin{aligned} M(x) &= -\frac{1}{x} \quad \Rightarrow \quad H_1(x) \equiv \int M(x)\, dx = -\int \frac{1}{x} dx = \ln|x| \\ N(y) &= \frac{1}{y^2} \quad \Rightarrow \quad H_2(y) \equiv \int N(y)\, dy = \int \frac{1}{y^2} dy = -\frac{1}{y} \end{aligned}$$

and so, (6) is equivalent to

$$-\ln|x| - \frac{1}{y} = C$$

Example 2, Cont'd

or

$$\frac{1}{y} = \ln|x| - C$$

or

$$y(x) = \frac{1}{\ln|x| - C} \tag{8}$$

This is our general solution; the functional form of every solution of (6). .

Example 2, Cont'd

Let's now find the particular solution that satisfies the initital condition (7).
This we do by substituting the general solution (8) into the initial condition (7) and solving for C.

$$2 = y(1) = \left(\frac{1}{\ln|x| - C} \right) \Bigg|_{x=1} = \frac{1}{\ln(1) - C} = \frac{1}{0 - C} = -\frac{1}{C}$$

The extreme sides of this last equation tells us that $C = -\frac{1}{2}$ when (7) is satisfied.
Thus,

$$y(x) = \frac{1}{\ln|x| + \frac{1}{2}}$$

is the solution to the differential equation (6) that satisfies the initial condition (7).

Exact Solutions of 1st Order ODEs: Results Thus Far

Standard Form of a 1st Order ODE

$$\frac{dy}{dx} = F(x,y) \qquad (*)$$

Methods for Special Cases:

1. $F(x,y) = f(x)$

   $$\text{Solution:} \quad y(x) = \int f(x)\,dx + C$$

   by the Fundamental Theorem of Calculus

2. $F(x,y) = -\frac{M(x)}{N(y)}$ Solution found by converting the differential equation to an algebraic equation

   $$\int M(x)\,dx + \int N(y)\,dy = C$$

   and solving this equation for y as a function of x.

The next special case: 1st Order Linear ODEs

We will next consider the case where the function F on the RHS of (*) is of the form

$$F(x, y) = -p(x)y + g(x)$$

where p and g are function of x alone.
In this case, the ODE (*) can be rewritten as

$$\frac{dy}{dx} + p(x)y = g(x) \tag{9}$$

So this form of $F(x, y)$ leads to a 1st Order Linear ODE.
Our goal now is to find solutions of differential equations of the form (9).

Homogeneous, 1st Order, Linear, ODEs

In the next lecture, I will derive a formula for the general solution of

$$\frac{dy}{dx} + p(x)y = g(x) \tag{9}$$

Today, however, we will just look at a special subcase such equations: where the function $g(x)$ on the RHS is 0.

ODEs of the form

$$\frac{dy}{dx} + p(x)y = 0 \tag{10}$$

are called **homogeneous**, linear 1st order, ODEs. The qualification "homogeneous", simply means that the function $g(x)$ that appears on the RHS of (9) is 0.

Thus, we begin our study of first order linear ODEs with the equations of the form (10)

Homogeneous, 1st Order, Linear, ODEs, Cont'd

Happily, we already have a method for solving this case. If we multiply both sides of (10) by $\frac{1}{y}$, we get

$$p(x) + \frac{1}{y}\frac{dy}{dx} = 0$$

which is a **separable ODE** with

$$\begin{aligned} M(x) &= p(x) \\ N(y) &= \frac{1}{y} \end{aligned}$$

Homogeneous, 1st Order, Linear, ODEs Cont'd

Let's then apply the technique for solving separable ODEs.
We have

$$\begin{aligned} H_1(x) &= \int M(x)\,dx = \int p(x)\,dx \\ H_2(y) &= \int N(y)\,dy = \int \frac{1}{y}dy = \ln|y| \end{aligned}$$

and so solutions of (10) must satisfy an algebraic equation of the form $H_1(x) + H_2(y) = C$, or

$$\int p(x)\,dx \; + \; \ln|y| = C$$

or

$$\ln|y| = C - \int p(x)\,dx$$

Homogeneous, 1st Order, Linear, ODEs Cont'd

or

$$\begin{aligned} y &= \exp\left(\ln\left|y\right|\right) = \exp\left[C - \int p\left(x\right)dx\right] \\ &= e^{C}\exp\left[-\int p\left(x\right)dx\right] \end{aligned}$$

If C is to be an arbitrary number, e^{C} will be just as arbitary. It is a common to practice to replace the arbitary constant factor e^{C} by a simpler constant factor A. Then we have

Theorem

The general solution to

$$\frac{dy}{dx} + p\left(x\right)y = 0$$

is given by

$$y\left(x\right) = A\exp\left[-\int p\left(x\right)dx\right]$$

Example

Find the general solution of

$$\frac{dy}{dx} + \frac{2}{x} y = 0$$

This ODE is of the form $y' + p(x)\, y = 0$ with $p(x) = \frac{2}{x}$.
Using the formula just derived we have, as the general solution,

$$\begin{aligned} y(x) &= A \exp\left[-\int p(x)\, dx\right] = A \exp\left[-\int \frac{2}{x} dx\right] \\ &= A \exp\left[-2 \ln |x|\right] \\ &= A \left(\exp\left(\ln |x|\right)\right)^{-2} \\ &= A(x)^{-2} \\ &= \frac{A}{x^2} \end{aligned}$$