Math 2233 - Lecture 4

Agenda:

1. Separable Equations: Theory
2. Examples of Solving Separable Equation
3. 1st Order Linear ODEs, Intro

Separable First Order ODEs

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I also gave a quick and dirty method for writing down some exact solutions; the so-called Mneumonic Method, whereby, one converts the original separable differential equation (1) to an equivalent algebraic equation.
The equivalent algebraic equation is then used to solve for y; and the resulting formula for y will yield a function of x that solves the original differential equation.

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$$\int N(y)\,dy = \int -M(x)\,dx \tag{3}$$
4. Let $H_1(x) \equiv \int M(x)\,dx$ and let $H_2(y) \equiv \int N(y)\,dy$, then (3) is equivalent to
$$H_1(x) + H_2(y) = 0 \tag{2}$$

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5. An exact solution of the original separable equation is found by solving (2) for y.

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Today, I'll show you the details of why this method works and how we find *all* of the solutions.

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This raises another question: How can a differential equation be related to an algebraic equation?

Digression: Implicit Differentiation

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$$x^2 + y^2 = 1$$

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$$\frac{d}{dx}\left(x^2 + (y(x))^2\right) = \frac{d}{dx}(1)$$

$$\implies \quad 2x + 2 * y(x)\frac{dy}{dx} = 0$$

Digression: Implicit Differentiation, Cont'd

Note how the Chain Rule

$$\frac{d}{dx}\left(h\left(y(x)\right)\right) = \frac{dh}{dy}\frac{dy}{dx}$$

was employed to compute $\frac{d}{dx}\left(y(x)\right)^2$. as $2y\frac{dy}{dx}$)

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We have thus **derived** a differential equation from the original algebraic equation.

It follows that solutions of the differential equation will also be solutions the original algebraic equation.

Constructing Separable ODEs by Implicit Differentiation

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where C is a constant.

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If we carry out implicit differentiation of this equation, we get

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If we carry out implicit differentiation of this equation, we get

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = 0$$

Note that the resulting differential equation is of the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

with

$$M(x) = H_1'(x) \qquad , \qquad N(y) = H_2'(y)$$

Thus, implicit differentiation of algebraic equation of the form (*) produces a separable ODE with the same solution set.

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Let us now think of these two equations as differential equations for $H_1(x)$ and $H_2(y)$. Both of these equations are readily solved using the Fundamental Theorem of Calculus (as we discussed at the beginning of this lecture)

Deriving an Algebraic Equation from a Separable ODE, Cont'd

$$
\begin{aligned}
M(x) &= H_1'(x) \quad \Longrightarrow \quad H_1(x) = \int M(x)\, dx + c_1 \\
N(y) &= H_2'(y) \quad \Longrightarrow \quad H_2(y) = \int N(y)\, dy + c_2
\end{aligned}
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Substituting these results into the original algebraic equation we get

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When C equals 0 we get the same algebraic equation that was produced by our (mathematically improper) Mneumonic Method. Moreover, we now see how to get more solutions from our Mneumonic method; for, in the present derivation, C is an arbitrary constant, not necessarily equal to 0.

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3. Solve the algebraic equation for y as a function of x and C.

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(4) is of Separable Form, with

$$M(x) = \frac{\cos(x)}{\sin(x)} \qquad , \qquad N(y) = \frac{1}{y}$$

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So (4) is equivalent to the following algebraic algebraic equation

$$H_1(x) + H_2(y) = C \quad \Rightarrow \quad \ln(\sin(x)) + \ln(y) = C$$

Solving this last equation of y, we get

$$\ln|y| = C - \ln(\sin(x))$$

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$$\ln|y| = C - \ln(\sin(x))$$

or, after exponentiating both sides,

$$\exp(\ln|y|) = \exp(C - \ln(\sin(x))) \tag{5}$$

Example 1, Cont'd

Next, we use some identities for exponential and logarithmic functions

$$
\begin{aligned}
\exp(C) &\equiv e^{C} \\
\exp(\ln|x|) &= x \\
\exp(A+B) &= \exp(A)\exp(B) \\
\exp(\lambda B) &= \exp(B)^{\lambda}
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to simplify both sides of (5) We find

$$LHS = \exp(\ln|y|) = y$$

and

$$\begin{aligned} RHS &= \exp(C - \ln(\sin(x))) \\ &= \exp(C)\exp(-\ln(\sin(x))) \\ &= \exp(C)(\exp(\ln(\sin(x))))^{-1} \\ &= e^{C}(\sin(x))^{-1} \\ &= \frac{e^{C}}{\sin(x)} \end{aligned}$$

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$$y(x) = \frac{e^C}{\sin(x)}$$

This is the functional form of the general solution to the differential equation.

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satisfying the initial condition

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Thus,

$$\begin{aligned} M(x) &= -\frac{1}{x} \quad \Rightarrow \quad H_1(x) \equiv \int M(x)\,dx = -\int \frac{1}{x}dx = \ln|x| \\ N(y) &= \frac{1}{y^2} \quad \Rightarrow \quad H_2(y) \equiv \int N(y)\,dy = \int \frac{1}{y^2}dy = -\frac{1}{y} \end{aligned}$$

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and so, (6) is equivalent to

$$-\ln|x| - \frac{1}{y} = C$$

Example 2, Cont'd

or

$$\frac{1}{y} = \ln|x| - C$$

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$$\frac{1}{y} = \ln|x| - C$$

or

$$y(x) = \frac{1}{\ln|x| - C} \tag{8}$$

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This is our general solution; the functional form of every solution of (6). .

Example 2, Cont'd

Let's now find the particular solution that satisfies the initital condition (7).

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Thus,

$$y(x) = \frac{1}{\ln|x| + \frac{1}{2}}$$

is the solution to the differential equation (6) that satisfies the initial condition (7).

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   and solving this equation for y as a function of x.

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We will next consider the case where the function F on the RHS of (*) is of the form

$$F(x, y) = -p(x)y + g(x)$$

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So this form of $F(x, y)$ leads to a 1st Order Linear ODE.
Our goal now is to find solutions of differential equations of the form (9).

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Thus, we begin our study of first order linear ODEs with the equations of the form (10)

Homogeneous, 1st Order, Linear, ODEs, Cont'd

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Homogeneous, 1st Order, Linear, ODEs Cont'd

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$$\begin{aligned}
y(x) &= A\exp\left[-\int p(x)\,dx\right] = A\exp\left[-\int \frac{2}{x}dx\right] \\
&= A\exp\left[-2\ln|x|\right] \\
&= A\left(\exp\left(\ln|x|\right)\right)^{-2} \\
&= A(x)^{-2} \\
&= \frac{A}{x^2}
\end{aligned}$$