Math 2233 - Lecture 5

Agenda: 1st Order ODEs:

1. Standard form $\frac{dy}{dx} + p(x)y = g(x)$
2. Case (i) : $p(x) = 0$
3. Case (ii): $g(x) = 0$
4. Case (iii): $g(x) = a$, a constant
5. The General Case

Solving 1st Order Linear ODEs

A **linear first order ordinary differential equation** is a differential equation of the form

$$a(x)\frac{dy}{dx} + b(x)y = c(x) \quad . \tag{1}$$

So long as $a(x) \neq 0$, this equation is equivalent to a differential equation of the form

$$y' + p(x)y = g(x) \tag{2}$$

where

$$y' = \frac{dy}{dx} \quad , \quad p(x) = \frac{b(x)}{a(x)} \quad , \quad g(x) = \frac{c(x)}{a(x)}$$

We shall refer to a differential equation (2) as the **standard form** of differential equation (1).

Our goal now is to develop a formula for the general solution of (2).

To acheive this goal, we shall first construct solutions for several special cases.

Then with the knowledge gained from these simpler examples, we will develop a general formula for the solution of **any** differential equation of the form (2).

Case (i): $p(x) = 0$, $g(x) =$ some function of x

In this case, we have

$$\frac{dy}{dx} = g(x) \tag{3}$$

and so we are looking for a function whose derivative is $g(x)$. Last week we showed that, for this situation, the Fundamental Theorem of Calculus yields the following general solution

$$y(x) = \int g(x)dx \ + \ C \tag{4}$$

where C is an arbitrary constant of integration.

Example

$$y' = 3\cos(4x) \tag{5}$$

$$\begin{aligned} \Rightarrow \quad y(x) &= \int 3\cos(4x)\,dx \quad + \quad C \\ &= \frac{3}{4}\sin(4x) + C \end{aligned}$$

So the general solution of (5) is

$$y(x) = \frac{3}{4}\sin(4x) + C$$

Case (ii): $g(x) = 0$, $p(x) =$ some function of x

In this case we are trying to solve a differential equation of the form

$$y' + p(x)y = 0 \quad . \tag{6}$$

If we divide both sides of (6) by y and reorder terms, we get

$$p(x) + \frac{1}{y}\frac{dy}{dx} = 0 \tag{7}$$

This equation is a **separable** 1st order differential equation; i.e., an ODE of the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Case (ii), Cont'd

Last week, we derived the following recipe for solving separable equations in the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

- Compute functions $H_1(x)$ and $H_2(y)$ as

$$H_1(x) = \int M(x)\,dx \quad , \quad H_2(y) = \int N(y)\,dy$$

- Then solve

$$H_1(x) + H_2(y) = C$$

  for y as a function of x and the constant C

Let's apply this procedure to the case at hand;

Case (ii), Cont'd

For the separable equation (7), we have

$$\begin{aligned} M(x) &= p(x) &\Longrightarrow \quad H_1(x) = \int p(x)\,dx \\ N(y) &= \frac{1}{y} &\Longrightarrow \quad H_2(y) = \ln|y| \end{aligned}$$

And so we need to solve

$$\int p(x)\,dx \; + \; \ln|y| = C$$

Solving this last equation for y yields

$$y = \exp\left[-\int^x p(x)\,dx \quad + \quad C\right]$$

or

$$y = e^C \exp\left[-\int^x p(x)\,dx\right]$$

Case (ii), Cont'd

We can tidy this up a little bit by replacing e^C, which is just an arbitrary constant, by an equivalent arbitrary constant A, to write

$$y(x) = A \exp\left[-\int^x p(x)\,dx\right]$$

Case (ii) Summary

The general solution of

$$y' + p(x) = 0$$

is given by

$$y = A \exp \left[- \int^{x} p(x)\, dx \right] \quad .$$

where A is an arbitrary constant.

Case (iii): $g(x) \neq 0$, $p(x) = a$, a constant

In this case, we have

$$\frac{dy}{dx} + ay = g(x) \quad .$$

To solve this equation we employ a trick.
Suppose we multiply both sides of this equation by e^{ax}:

$$e^{ax}y' + ae^{ax}y = e^{ax}g(x).$$

Notice that the right hand side is $\frac{d}{dx}(e^{ax}y)$ (via the product rule for differentiation) We thus have

$$\frac{d}{dx}(e^{ax}y) = e^{ax}g(x)$$

We now take anti-derivatives of both sides to get

$$e^{ax}y = \int^{x} e^{ax}g(x)\,dx \quad + \quad C$$

Case (iii), Cont'd

or

$$y(x) = \frac{1}{e^{ax}} \int^x e^{ax} g(x)\, dx \quad + \quad Ce^{-ax} \quad .$$

Thus, the general solution to

$$y' + ay = g(x)$$

is given by

$$y(x) = \frac{1}{e^{ax}} \int^x e^{ax} g(x)\, dx \quad + \quad Ce^{-ax}$$

where C is an arbitrary constant.

Example

$$y' - 2y = x^2 e^{2x}$$

This equation is of type (iii) with

$$\begin{aligned} a &= -2 \\ g(x) &= x^2 e^{2x} \end{aligned} \quad .$$

So we multiply both sides by e^{-2x} to get

$$\frac{d}{dx}\left(e^{-2x} y\right) = e^{-2x}\left(y' - 2y\right) = e^{-2x}\left(x^2 e^{2x}\right) = x^2$$

Integrating both sides with respect to x, and employing the Fundamental Theorem of Calculus on the left yields

$$e^{-2x} y = \frac{1}{3} x^3 + C$$

or

$$y = \frac{1}{3} x^3 e^{2x} + C e^{2x} \quad .$$

Example, Cont'd

Let us now confirm that this is a solution

$$\begin{aligned} y' &= x^2e^{2x} + \frac{2}{3}x^3e^{2x} + 2Ce^{2x} \\ -2y &= -\frac{2}{3}x^3e^{2x} - 2Ce^{2x} \end{aligned}$$

so

$$y' - 2y = x^2e^{2x}$$

□

1st Order Linear ODEs : The General Case

We are now ready to handle ODEs of the form

$$y' + p(x)y = g(x) \tag{8}$$

with $p(x)$ and $g(x)$ are arbitrary functions of x.

Note: This case includes all the preceding cases of linear 1st order ODEs.

We shall construct a solution of this equation in a manner similar to case when $p(x)$ is a constant. We will first will try find a multiplying function $\mu(x)$ (analogous to our use of e^{ax} in the preceding case) satisfying

$$\mu(x)\left(y' + p(x)y\right) = \frac{d}{dx}\left(\mu(x)y\right) \tag{9}$$

If we had such a function $\mu(x)$, we could multiply (8) by $\mu(x)$ to obtain

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x)$$

General Case of 1st Order Linear ODEs, Cont'd

Since the left hand side of this last equation is a pure derivative, it is readily integrated.
Integrating both sides of

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x)$$

yields

$$\mu(x)y = \int \mu(x)g(x)\,dx + C$$

or

$$y = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx \; + \; \frac{C}{\mu(x)} \tag{11}$$

and so we would end up with a closed formula for the solution.

General Case of 1st Order Linear ODEs, Cont'd

It thus remains to find a suitable multiplier function $\mu(x)$ that satisfies

$$\frac{d}{dx}(\mu(x)y) = \mu(x)\left(y' + p(x)\,y\right)$$

so that the computation just outlined can proceed. This will certainly be true if

$$\frac{d}{dx}\mu(x) = p(x)\mu(x) \quad . \tag{12}$$

For then

$$\frac{d}{dx}(\mu(x)y) = \mu(x)y' + \left(\frac{d}{dx}\mu(x)\right)y = \mu(x)y' + p(x)\,\mu(x)y$$

But (12) is another first order, linear, differential equation of type (iii); (This time, however, our unknown function is $\mu(x)$.)
As in the type (iii) case before, we recast (12) as a separable equation

$$-p(x) + \frac{1}{\mu}\frac{d\mu}{dx} = 0$$

in order to solve it:

Applying our method for separable equations to

$$-p(x) + \frac{1}{\mu}\frac{d\mu}{dx} = 0$$

we find

$$\begin{aligned} M(x) &= -p(x) \quad \Longrightarrow \quad H_1(x) = \int M(x)\,dx = \int -p(x)\,dx \\ N(\mu) &= \frac{1}{\mu} \quad \Longrightarrow \quad H_2(\mu) = \int N(y)\,dy = \ln(\mu) \end{aligned}$$

$$H_1(x) + H_2(\mu) = C \quad \Longrightarrow \quad -\int p(x)\,dx \; + \; ln(\mu) = C$$

Solving this last equation for μ yields

$$\begin{aligned} \mu(x) &= \exp\left(\int p(x)dx + C\right) \\ &= A\exp\left(\int p(x)\,dx\right) \qquad \text{(where } A \equiv e^C\text{)} \end{aligned}$$

So a suitable function $\mu(x)$ is

$$\mu(x) = \exp\left(\int p(x)\, dx\right)$$

I set the constant $A = 1$, because we don't need all the solutions of

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)\left(y' + p\left(x\right)y\right)$$

Any solution will do.

Solving 1st Order Linear ODEs: The General Case

A 1st order linear ODE in standard form

$$y' + p(x)y = g(x) \tag{13}$$

can be solved by the following procedure.

1. Calculate the "**integrating factor**" $\mu(x)$

$$\mu(x) = exp\left[\int p(x)dx\right]$$

2. This function has the property $\mu(x)\left(y' + p(x)y\right) = \frac{d}{dx}\left(\mu(x)y\right)$ and so after we multiplying both sides of (13) by $\mu(x)$ we get

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x)$$

3. Integrating both sides yields

$$\mu(x)\,y = \int \mu(x)\,g(x)\,dx + C$$

Solving 1st Order Linear ODEs: The General Case, Cont'd

4. And then finally we solve

$$\mu(x)\, y = \int \mu(x)\, g(x)\, dx + C$$

for y to get

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)\, g(x)\, dx \quad + \quad \frac{C}{\mu(x)}$$

Summary: Linear 1st Order ODEs

Theorem

The general solution to

$$y' + p(x)y = g(x)$$

is given by

$$y\left(x\right) = \frac{1}{\mu\left(x\right)}\int \mu\left(x\right)g\left(x\right)dx \quad + \quad \frac{C}{\mu\left(x\right)}$$

where

$$\mu(x) = exp\left[\int p(x)dx\right]$$

Example

$$xy' + 2y = \sin(x) \tag{14}$$

Putting this equation in standard form requires we set

$$\begin{aligned} p(x) &= \tfrac{2}{x} \\ g(x) &= \tfrac{\sin(x)}{x} \end{aligned}$$

Now

$$\int p(x)\,dx = \int \frac{2}{x} dx = 2\ln(x) = \ln\left(x^2\right),$$

so

$$\begin{aligned} \mu(x) &= \exp\left[\int^x p(x)\,dx\right] \\ &= \exp\left[\ln\left(x^2\right)\right] \\ &= x^2 \end{aligned}$$

Example, Cont'd

Hence

$$
\begin{aligned}
y(x) &= \frac{1}{\mu(x)} \int \mu(x) g(x)\, dx + \frac{C}{\mu(x)} \\
&= \frac{1}{x^2} \int (x)^2 \frac{\sin(x)}{x}\, dx + \frac{C}{x^2} \\
&= \frac{1}{x^2} \int x \sin(x)\, dx + \frac{C}{x^2}
\end{aligned}
$$

Now

$$\int x \sin(x)\, dx$$

can be integrated by parts. Set

$$u = x \quad , \quad dv = \sin(x) dx$$

Example, Cont'd

Then

$$du = dx \qquad , \qquad v = \int dv = -\cos(x)$$

and the integration by parts formula,

$$\int u dv = uv - \int v du \quad ,$$

tells us that

$$\begin{aligned} \int x \sin(x)\, dx &= -x\cos(x) + \int \cos(x)\, dx \\ &= -x\cos(x) + \sin(x) \quad . \end{aligned}$$

Therefore, we have as a general solution of (14),

$$\begin{aligned} y(x) &= \frac{1}{x^2}\left(-x\cos(x) + \sin(x)\right) + \frac{C}{x^2} \\ &= \frac{1}{x^2}\sin(x) - \frac{1}{x}\cos(x) + \frac{C}{x^2} \quad . \end{aligned}$$

Initial Value Problems for 1st Order Linear ODEs

Consider

$$\begin{aligned} x^2 y' + 3xy &= 1 \\ y(1) &= 1 \end{aligned}$$

This is a 1st order linear ODE with an initial condition and so we expect a unique solution. Here is how we can find it.
Step 1: Put the Diff E in standard form (for a 1st order linear ODE): $y' + p(x) y = g(x)$

$$\frac{1}{x^2}\left(x^2 y' + 3xy\right) = \frac{1}{x_2}(1) \quad \Rightarrow \quad y' + \frac{3}{x} y = \frac{1}{x^2}$$

So

$$p(x) = \frac{3}{x} \quad , \quad g(x) = \frac{1}{x^2}$$

Initial Value Problems, Cont'd

Step 2: Calculate the integrating factor $\mu(x)$:

$$\begin{aligned} \mu(x) &= \exp\left[\int p(x)\, dx\right] \\ &= \exp\left[\int \frac{3}{x} dx\right] = \exp\left[3 \ln |x|\right] = x^3 \end{aligned}$$

where I used the identity

$$\exp(\lambda \ln |x|) = x^{\lambda}$$

Initial Value Problems, Cont'd

Step 3: Calculate the general solution

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)} \int \mu(x) g(x)\, dx \quad + \quad \frac{C}{\mu(x)} \\ &= \frac{1}{x^3} \int (x^3) \left(\frac{1}{x^2}\right) dx \quad + \quad \frac{C}{x^3} \\ &= \frac{1}{x^3} \left(\frac{1}{2}x^2\right) + \frac{C}{x^3} \\ &= \frac{1}{2x} + \frac{C}{x^3} \end{aligned}$$

Step 4: Impose the intitial condition on the general solution

$$1 = y(1) = \left.\left(\frac{1}{2x} + \frac{C}{x^3}\right)\right|_{x=1} = \frac{1}{2} + C$$

The extreme sides of this equation tell us that

$$C = \frac{1}{2}$$

Initial Value Problems, Cont'd

Step 5: Substitute the correct value for C into the general solution to get the solution satisfying the initial condition

$$y(x) = \frac{1}{2x} + \frac{\frac{1}{2}}{x^3} = \frac{1}{2x} + \frac{1}{2x^3}$$

An Alternative Procedure for Solving Initial Value Problems for 1st Order Linear ODEs

Theorem

The unique solution to

$$\begin{aligned} y' + p(x)\, y' &= g(x) \\ y(x_0) &= y_0 \end{aligned}$$

can be obtained as follows:

- *Compute*

$$\mu_0(x) = \exp\left[\int_{x_0}^{x} p(s)\, ds\right]$$

- *Then compute*

$$y(x) = \frac{1}{\mu_0(x)} \int_{x_0}^{x} \mu_0(s)\, g(s)\, ds \quad + \quad \frac{y_0}{\mu_0(x)}$$

Comparing Formulas for Solutions of $y' + p(x)y = g(x)$

1. The General Solution

$$\mu(x) = \exp\left[\int p(x)\,dx\right]$$

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)\,g(x)\,dx + \frac{C}{\mu(x)}$$

2. The Unique Solution satisfying $y(x_0) = y_0$

$$\mu_0(x) = \exp\left[\int_{x_0}^{x} p(s)\,ds\right]$$

$$y(x) = \frac{1}{\mu_0(x)}\int_{x_0}^{x} \mu_0(s)\,g(s)\,ds + \frac{y_0}{\mu_0(x)}$$