Math 2233 - Lecture 5

Agenda: 1st Order ODEs:

1. Standard form $\frac{dy}{dx} + p(x)y = g(x)$
2. Case (i) : $p(x) = 0$
3. Case (ii): $g(x) = 0$
4. Case (iii): $g(x) = a$, a constant
5. The General Case

Solving 1st Order Linear ODEs

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We shall refer to a differential equation (2) as the **standard form** of differential equation (1).

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Then with the knowledge gained from these simpler examples, we will develop a general formula for the solution of **any** differential equation of the form (2).

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$$y(x) = \int g(x)dx \; + \; C \tag{4}$$

where C is an arbitrary constant of integration.

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So the general solution of (5) is

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This equation is a **separable** 1st order differential equation; i.e., an ODE of the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

Case (ii), Cont'd

Last week, we derived the following recipe for solving separable equations in the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

- Compute functions $H_1(x)$ and $H_2(y)$ as

$$H_1(x) = \int M(x)\,dx \quad , \quad H_2(y) = \int N(y)\,dy$$

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Let's apply this procedure to the case at hand;

Case (ii), Cont'd

For the separable equation (7), we have

$$\begin{aligned} M(x) &= p(x) \quad \Longrightarrow \quad H_1(x) = \int p(x)\,dx \\ N(y) &= \frac{1}{y} \quad \Longrightarrow \quad H_2(y) = \ln|y| \end{aligned}$$

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or

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Case (ii) Summary

The general solution of

$$y' + p(x) = 0$$

is given by

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where A is an arbitrary constant.

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$$\frac{d}{dx}(e^{ax}y) = e^{ax}g(x)$$

We now take anti-derivatives of both sides to get

$$e^{ax}y = \int^{x} e^{ax}g(x)\,dx \quad + \quad C$$

Case (iii), Cont'd

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$$y(x) = \frac{1}{e^{ax}} \int^{x} e^{ax} g(x)\, dx \quad + \quad Ce^{-ax} \quad .$$

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Let us now confirm that this is a solution

$$\begin{aligned} y' &= x^2 e^{2x} + \frac{2}{3} x^3 e^{2x} + 2Ce^{2x} \\ -2y &= -\frac{2}{3} x^3 e^{2x} - 2Ce^{2x} \end{aligned}$$

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with $p(x)$ and $g(x)$ are arbitrary functions of x.

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If we had such a function $\mu(x)$, we could multiply (8) by $\mu(x)$ to obtain

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)g(x)$$

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and so we would end up with a closed formula for the solution.

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so that the computation just outlined can proceed.

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$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)\left(y' + p\left(x\right)y\right)$$

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$$-p(x) + \frac{1}{\mu}\frac{d\mu}{dx} = 0$$

in order to solve it:

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we find

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$$H_1(x) + H_2(\mu) = C \quad \Longrightarrow \quad -\int p(x)\,dx \; + \; ln(\mu) = C$$

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$$H_1(x) + H_2(\mu) = C \implies -\int p(x)\,dx \;+\; ln(\mu) = C$$

Solving this last equation for μ yields

$$\begin{aligned} \mu(x) &= \exp\left(\int p(x)dx + C\right) \\ &= A\exp\left(\int p(x)\,dx\right) \qquad (\text{where } A \equiv e^C) \end{aligned}$$

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$$H_1(x) + H_2(\mu) = C \implies -\int p(x)\,dx \; + \; ln(\mu) = C$$

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I set the constant $A = 1$, because we don't need all the solutions of

$$\frac{d}{dx}\left(\mu(x)y\right) = \mu(x)\left(y' + p\left(x\right)y\right)$$

Any solution will do.

Solving 1st Order Linear ODEs: The General Case

A 1st order linear ODE in standard form

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3. Integrating both sides yields

$$\mu\left(x\right)y = \int \mu\left(x\right)g\left(x\right)dx + C$$

Solving 1st Order Linear ODEs: The General Case, Cont'd

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The general solution to

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so

$$\begin{aligned} \mu(x) &= \exp\left[\int^x p(x)\,dx\right] \\ &= \exp\left[\ln\left(x^2\right)\right] \\ &= x^2 \end{aligned}$$

Example, Cont'd

Hence

$$
\begin{aligned}
y(x) &= \frac{1}{\mu(x)} \int \mu(x) g(x)\, dx + \frac{C}{\mu(x)} \\
&= \frac{1}{x^2} \int (x)^2 \frac{\sin(x)}{x}\, dx + \frac{C}{x^2} \\
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Therefore, we have as a general solution of (14),

$$\begin{aligned} y(x) &= \frac{1}{x^2}\left(-x\cos(x) + \sin(x)\right) + \frac{C}{x^2} \\ &= \frac{1}{x^2}\sin(x) - \frac{1}{x}\cos(x) + \frac{C}{x^2} \quad . \end{aligned}$$

Initial Value Problems for 1st Order Linear ODEs

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So

$$p(x) = \frac{3}{x} \quad , \quad g(x) = \frac{1}{x^2}$$

Initial Value Problems, Cont'd

Step 2: Calculate the integrating factor $\mu(x)$:

$$\begin{aligned}
\mu(x) &= \exp\left[\int p(x)\,dx\right] \\
&= \exp\left[\int \frac{3}{x}dx\right] = \exp\left[3\ln|x|\right] = x^3
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where I used the identity

$$\exp(\lambda \ln|x|) = x^\lambda$$

Initial Value Problems, Cont'd

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$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\int \mu(x)\,g(x)\,dx + \frac{C}{\mu(x)} \\ &= \frac{1}{x^3}\int \left(x^3\right)\left(\frac{1}{x^2}\right)dx + \frac{C}{x^3} \\ &= \frac{1}{x^3}\left(\frac{1}{2}x^2\right) + \frac{C}{x^3} \\ &= \frac{1}{2x} + \frac{C}{x^3} \end{aligned}$$

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The extreme sides of this equation tell us that

$$C = \frac{1}{2}$$

Initial Value Problems, Cont'd

Step 5: Substitute the correct value for C into the general solution to get the solution satisfying the initial condition

$$y(x) = \frac{1}{2x} + \frac{\frac{1}{2}}{x^3} = \frac{1}{2x} + \frac{1}{2x^3}$$

An Alternative Procedure for Solving Initial Value Problems for 1st Order Linear ODEs

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The unique solution to

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- *Then compute*

$$y(x) = \frac{1}{\mu_0(x)} \int_{x_0}^{x} \mu_0(s)\, g(s)\, ds \quad + \quad \frac{y_0}{\mu_0(x)}$$

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2. The Unique Solution satisfying $y(x_0) = y_0$

$$\begin{aligned} \mu_0(x) &= \exp\left[\int_{x_0}^{x} p(s)\,ds\right] \\ y(x) &= \frac{1}{\mu_0(x)}\int_{x_0}^{x} \mu_0(s)\,g(s)\,ds \quad + \quad \frac{y_0}{\mu_0(x)} \end{aligned}$$