Math 2233 - Lecture 6

Agenda

1. Comments on Homework Problems
2. Examples of 1st Order Linear ODEs
3. Exact Equations

HW2 : Problem 2.2.21

Solve the initial value problem

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\cos(\theta)}{y^3+1} \quad , \qquad y(\pi) = 1 \tag{1}$$

Note that for this ODE

- θ is the underlying variable
- y is the unknown function

HW Example, Cont'd

First, we recast the equation into to the manifestly separable form $M(\theta) + N(y)\frac{dy}{d\theta} = 0$. This we do by multiplying both sides by $\theta\frac{y^3+1}{y}$ and putting everything on the left hand side.

$$-\theta\cos(\theta) + \left(y^2 + \frac{1}{y}\right)\frac{dy}{d\theta} = 0$$

$$\begin{aligned}
M(\theta) &= -\theta\cos(\theta) \\
\Rightarrow \quad H_1(\theta) &= \int M(\theta)\,d\theta = -\int \theta\cos(\theta)\,d\theta = -\theta\sin(\theta) - \cos(\theta
\\
N(y) &= y^2 + \frac{1}{y} \\
\Rightarrow \quad H_2(y) &= \int N(y)\,dy = \int \left(y^2 + \frac{1}{y}\right)dy = \frac{1}{3}y^3 + \ln|y|
\end{aligned}$$

HW Example, Cont'd

The differential equation (being separable) is then equivalent to the following algebraic equation

$$-\theta \sin(\theta) - \cos(\theta) + \frac{1}{3}y^3 + \ln|y| = C \tag{2}$$

Unfortunately, this equation can not be solved for y as a function of x and C
(it is what's called a **transcendental equation**; a valid equation that cannot be solved by algebraic methods).
We call equation (2) the **implicit solution** to the ODE in equation (1).
Yet, we can still employ (2) and the initial condition to fix a value for the constant C.
When $\theta = \pi$, we must have $y = 1$ and so (2) requires

$$-\pi \sin(\pi) - cos(\pi) + \frac{1}{3} + 0 = C \quad \Rightarrow \quad C = \frac{4}{3}$$

HW Example, Cont'd

Thus, the implicit solution to the initial value problem (ODE and initial condition) is

$$-\theta \sin(\theta) - \cos(\theta) + \frac{1}{3}y^3 + \ln|y| = \frac{4}{3}$$

Review of Linear ODEs with Initial Values

In order to review the results of the last lecture, let's work out the solution to

$$\begin{aligned} \frac{dy}{dx} - \frac{1}{x} y &= x e^{x} \\ y(1) &= e - 1 \end{aligned}$$

two different ways.

Method 1: Find the General Solution and then figure out the right value for the constant C

Theorem

The general solution to

$$y' + p(x)\, y = g(x)$$

is found by computing

$$\mu(x) \equiv \exp\left[\int p(x)\, dx\right]$$

and then

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)\, g(x)\, dx \quad + \quad \frac{C}{\mu(x)}$$

1st Method for Linear IVPs, Cont'd

In the case at hand,

$$p(x) = -\frac{1}{x} \quad , \quad g(x) = xe^x$$

and so

$$\mu(x) = \exp\left[-\int \frac{1}{x} dx\right] = \exp\left[-\ln|x|\right] = x^{-1} = \frac{1}{x}$$

and

$$\begin{aligned} y(x) &= \frac{1}{\frac{1}{x}} \int \frac{1}{x}\left(xe^x\right) dx + \frac{C}{\frac{1}{x}} \\ &= x \int e^x dx + Cx \\ &= xe^x + Cx \end{aligned}$$

So our general solution is

$$y(x) = xe^x + Cx$$

1st Method for Linear IVPs, Cont'd

Now let's impose the initial condition $y(1) = e - 1$

$$e - 1 = y(1) = e + C \qquad \Rightarrow \qquad C = -1$$

and so the solution to the initial value problem is

$$y(x) = xe^{x} - x$$

Method 2: Use the formula for the solution of a 1st order, linear, initial value problem

Theorem

The unique solution to

$$y' + p(x)\, y = g(x) \quad , \quad y(x_0) = y_0$$

is found by computing

$$\mu_0(x) \equiv \exp\left[\int_{x_0}^{x} p(s)\, ds\right]$$

and then

$$y(x) = \frac{1}{\mu_0(x)} \int_{x_0}^{x} \mu_0(s)\, g(s)\, dx \quad + \quad \frac{y_0}{\mu_0(x)}$$

We still have

$$p(x) = -\frac{1}{x} \quad , \quad g(x) = xe^{x}$$

Method 2 for Linear IVPs, Cont'd

So we can begin calculating:

$$\begin{aligned}\mu_0(x) &= \exp\left[\int_1^x \left(-\frac{1}{s}\right) dx\right] \\ &= \exp\left[-\ln[s]\big|_{s=1}^{s=x}\right] = \exp(-\ln|x|+0) = \frac{1}{x}\end{aligned}$$

$$\begin{aligned}y(x) &= \frac{1}{\frac{1}{x}}\int_1^x \left(\frac{1}{s}\right)(se^s)\,ds \quad + \quad \frac{e-1}{\frac{1}{x}} \\ &= x\int_1^x e^s ds \quad + \quad (e-1)x \\ &= x(e^x-e)+(e-1)x \\ &= xe^x - x\end{aligned}$$

Thus,

$$y(x) = xe^x - x$$

Exact Equations

The next special type of 1st order ODEs we'll consider is that of **Exact Equations**.

The method for exact equations is a generalization of the method we used for *separable equations*. The basic idea will again be to solve the differential equation by solving an equivalent algebraic equation.

Recall that we solved separable equations of the form

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{3}$$

by solving instead

$$H_1(x) + H_2(y) = C \tag{4}$$

where

$$H_1(x) = \int M(x)\,dx \quad , \quad H_2(y) = \int N(y)\,dy \tag{5}$$

This worked because if $H_1(x)$ and $H_2(y)$ are defined by (5) then we can derive (3) from (4) via implicit differentiation.

Exact Equation, Cont'd

Equation (4), however, is not the most general form for an algebraic equation relating x and y.

In fact, we might call (4) a **separable algebraic equation**, since the x-dependent terms are separate from the y-dependent terms in the equation.

A more general algebraic relationship between x and y can be expressed as

$$\Phi(x, y) = C \tag{6}$$

(and then (4) is just a special case of (6)).

I'll now derive a differential equation from (6) using implicit differentiation.

But first, we'll need a generalization of the chain rule.

Digression: The Chain Rule for Functions of 2 variables

For functions of a single variable, the Chain Rule is

$$\frac{d}{dx}\left(f\left(y\left(x\right)\right)\right) = \frac{df}{dy}\frac{dy}{dx}$$

If s, t are functions of x and y, then

$$\begin{aligned}
\frac{\partial}{\partial x}F\left(s\left(x,y\right),t\left(x,y\right)\right) &= \frac{\partial F}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial}{\partial t}\frac{\partial t}{\partial x}\\
\frac{\partial F}{\partial y}F\left(s\left(x,y\right),t\left(x,y\right)\right) &= \frac{\partial F}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial F}{\partial t}\frac{\partial t}{\partial y}
\end{aligned}$$

Exact Equations, Cont'd

Now suppose we carry out implicit differentiation of (6), from our multi-variable chain rule we have

$$\frac{d}{dx}\left(\Phi\left(x, y\left(x\right)\right)\right) = \frac{d}{dx}\left(C\right) \quad \Rightarrow \quad \frac{\partial \Phi}{\partial x}\frac{dx}{dx} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = 0 \tag{7}$$

or

$$\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = 0 \tag{8}$$

N.B., The term $\frac{\partial \Phi}{\partial x}$ and the factor $\frac{\partial \Phi}{\partial y}$ will, in general, depend on both x and y.

Exact Equations, Cont'd

And so (8) will have the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{9}$$

But there is also an additional hidden condition on the functions $M(x,y)$ and $N(x,y)$. Since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{\partial N}{\partial x}$$

Thus, if (9) is to be derivable from an algebraic equation of the form $\Phi(x,y) = C$, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations, Cont'd

Definition

A differential equation

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0$$

is called **exact**, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem

Suppose

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{10}$$

is exact (i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$). Then (10) is has the same solutions as an algebraic equation of the form

$$\Phi(x,y) = C \tag{11}$$

with $\Phi(x,y)$ determined (up to a constant) by the conditions

$$\frac{\partial \Phi}{\partial x} = M(x,y) \quad , \quad \frac{\partial \Phi}{\partial y} = N(x,y) \tag{12}$$

Of course, this immediately raises the question of exactly how $\Phi(x,y)$ is determined by (12).

To answer this question, we just need the 2-variable version of the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus for functions of 2 variables

Theorem

The most general solution of

$$\frac{\partial \Phi}{\partial x} = M(x, y)$$

is

$$\Phi(x, y) = \int M(x, y)\, \partial x \quad + \quad h_1(y)$$

where $h_1(y)$ is an arbitrary function depending only on y.
Similiarly, the most general solution of

$$\frac{\partial \Phi}{\partial y} = N(x, y)$$

is

$$\Phi(x, y) = \int N(x, y)\, \partial y \quad + \quad h_2(x)$$

Exact Equation Example

I'll now do an explicit example so that you see how this works.

Consider

$$\left(2xy^2+1\right)+\left(2x^2y\right)\frac{dy}{dx}=0 \tag{13}$$

For this equation we have

$$\begin{aligned} M(x,y) &= 2xy^2+1 \\ N(x,y) &= 2x^2y \end{aligned}$$

Noting that

$$\frac{\partial M}{\partial y}=4xy=\frac{\partial N}{\partial x}$$

we confirm that the differential equation is exact.

Exact Equations Example, Cont'd

The theorem about exact equations tells that (13) will have the same solutions as

$$\Phi(x, y) = C$$

with $\Phi(x, y)$ determined by

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= M(x, y) = 2xy^2 + 1 \\ \frac{\partial \Phi}{\partial y} &= N(x, y) = 2x^2 y \end{aligned}$$

Exact Equations Example, Cont'd

Applying our 2-variable Fundamental Theorem of Calculus

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= 2xy^2 + 1 \\ \Rightarrow \quad \Phi(x, y) &= \int \left(2xy^2 + 1\right) \partial x + h_1(y) \\ &= x^2y^2 + x + h_1(y) \end{aligned}$$

$$\begin{aligned} \frac{\partial \Phi}{\partial y} &= 2x^2y \\ \Rightarrow \quad \Phi(x, y) &= \int \left(2x^2y\right) \partial y + h_2(x) \\ &= x^2y^2 + h_2(x) \end{aligned}$$

Exact Equations Example, Cont'd

Now we have two separate equations for $\Phi(x, y)$

$$\begin{aligned} \Phi(x, y) &= x^2 y^2 + x + h_1(y) \\ \Phi(x, y) &= x^2 y^2 + h_2(x) \end{aligned}$$

These do not agree automatically. However, $h_1(y)$ and $h_2(x)$ are arbitrary functions that we can adjust to make these two expressions for $\Phi(x, y)$ agree with each other. If we set

$$h_1(y) = 0 \quad \text{and} \quad h_2(x) = x$$

Then both equations say

$$\Phi(x, y) = x^2 y^2 + x$$

Exact Equations Example, Cont'd

Having found the correct $\Phi(x, y)$, we can now solve (13) by solving instead

$$C = \Phi(x, y) = x^2y^2 + x$$

or

$$y(x) = \pm\sqrt{\frac{C - x}{x^2}} \tag{14}$$

(14) will thus be the general solution to the exact equation (13).

Summary: Solving Exact Equations

An exact differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{15}$$

with

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{16}$$

can be solved as follows:

0. Confirm the exactness condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is true.
1. If the equation is exact, then the differential equation is equivalent to an algebraic equation of the form

$$\Phi(x,y) = C$$

Summary: Solving Exact Equations, Cont'd

3. $\Phi(x, y)$ is determined by calculating

$$\begin{aligned}\Phi_1(x, y) &= \int M(x, y)\,\partial x + h_1(y) \\ \Phi_2(x, y) &= \int N(x, y)\,\partial y + h_2(x)\end{aligned}$$

   and then adjusting the arbitrary functions $h_1(y)$ and $h_2(x)$ so that $\Phi_1(x, y) = \Phi_2(x, y)$

4. We then set $\Phi(x, y) = \Phi_1(x, y) \quad (= \Phi_2(x, y))$ and solve (if possible)

$$\Phi(x, y) = C \tag{17}$$

   for y as a function of x and C

Summary: Solving Exact Equations, Cont'd

It sometimes happens that the algebraic equation

$$\Phi(x, y) = C \tag{17}$$

ends up being a transcendental equation which can not be explicitly solved for y.
In such cases, one just stops at equation (17) and refers to equation (17) as the **implicit solution** of (15).

Exact Equation Example 2

Consider

$$x + 2y + (2x + y)\frac{dy}{\partial x} = 0 \tag{18}$$

For this differential equation, we have

$$\begin{aligned} M(x, y) &= x + 2xy \\ N(x, y) &= x + y \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial M}{\partial y} &= 2 \\ \frac{\partial N}{\partial x} &= 2 \end{aligned}$$

and so the equation **is exact**.

Exact Equations Example 2, Cont'd

We now try to find the function $\Phi(x, y)$ so that solutions of (18) can be found by solving $\Phi(x, y) = C$

$$\begin{aligned} \Phi_1(x, y) &= \int M(x, y)\,\partial x + h_1(y) = \int (x + 2y)\,\partial x + h_1(y) \\ &= \frac{1}{2}x^2 + 2xy + h_1(y) \\ \Phi_2(x, y) &= \int N(x, y)\,\partial y + h_2(x) = \int (2x + y)\,\partial y + h_2(x) \\ &= 2xy + \frac{1}{2}y^2 + h_2(x) \end{aligned}$$

To get $\Phi_1(x, y) = \Phi_2(x, y)$, we need to set

$$\begin{aligned} h_1(y) &= \frac{1}{2}y^2 \\ h_2(x) &= \frac{1}{2}x^2 \end{aligned}$$

Exact Equations Example 2, Cont'd

Thus, with these choices for $h_1(y)$ and $h_2(x)$

$$\Phi(x, y) = \frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2$$

Exact Equations Example 2, Cont'd

The last step is to set $\Phi(x, y)$ equal to a constant C and solve for y :

$$\frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2 = C$$

$$\Rightarrow y = \frac{-2x \pm \sqrt{(2x)^2 - 4\left(\frac{1}{2}\right)\left(\frac{1}{2}x^2 - C\right)}}{2\left(\frac{1}{2}\right)}$$

where I have applied the Quadratic Formula

$$ay^2 + by + c = 0 \quad \Rightarrow \quad y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$