Math 2233 - Lecture 6

Agenda

1. Comments on Homework Problems
2. Examples of 1st Order Linear ODEs
3. Exact Equations

HW2 : Problem 2.2.21

Solve the initial value problem

$$\frac{1}{\theta}\frac{dy}{d\theta} = \frac{y\cos(\theta)}{y^3+1} \quad , \quad y(\pi) = 1 \tag{1}$$

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$$-\theta\cos(\theta) + \left(y^2 + \frac{1}{y}\right)\frac{dy}{d\theta} = 0$$

$$\begin{aligned} M(\theta) &= -\theta\cos(\theta) \\ \Rightarrow \quad H_1(\theta) &= \int M(\theta)\,d\theta = -\int \theta\cos(\theta)\,d\theta = -\theta\sin(\theta) - \cos(\theta \\ N(y) &= y^2 + \frac{1}{y} \\ \Rightarrow \quad H_2(y) &= \int N(y)\,dy = \int \left(y^2 + \frac{1}{y}\right)dy = \frac{1}{3}y^3 + \ln|y| \end{aligned}$$

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(it is what's called a **transcendental equation**; a valid equation that cannot be solved by algebraic methods).
We call equation (2) the **implicit solution** to the ODE in equation (1).
Yet, we can still employ (2) and the initial condition to fix a value for the constant C.
When $\theta = \pi$, we must have $y = 1$ and so (2) requires

$$-\pi \sin(\pi) - \cos(\pi) + \frac{1}{3} + 0 = C \quad \Rightarrow \quad C = \frac{4}{3}$$

HW Example, Cont'd

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Thus, the implicit solution to the initial value problem (ODE and initial condition) is

$$-\theta \sin(\theta) - \cos(\theta) + \frac{1}{3}y^3 + \ln|y| = \frac{4}{3}$$

Review of Linear ODEs with Initial Values

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In order to review the results of the last lecture, let's work out the solution to

$$\begin{aligned} \frac{dy}{dx} - \frac{1}{x}y &= xe^x \\ y(1) &= e - 1 \end{aligned}$$

two different ways.

Method 1: Find the General Solution and then figure out the right value for the constant C

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$$\mu(x) \equiv \exp\left[\int p(x)\, dx\right]$$

and then

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)\, g(x)\, dx \quad + \quad \frac{C}{\mu(x)}$$

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and

$$\begin{aligned} y(x) &= \frac{1}{\frac{1}{x}} \int \frac{1}{x}(xe^x)\, dx \quad + \quad \frac{C}{\frac{1}{x}} \\ &= x \int e^x dx \quad + \quad Cx \\ &= xe^x + Cx \end{aligned}$$

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So our general solution is

$$y(x) = xe^x + Cx$$

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Now let's impose the initial condition $y(1) = e - 1$

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$$e - 1 = y(1) = e + C \quad \Rightarrow \quad C = -1$$

and so the solution to the initial value problem is

$$y(x) = xe^{x} - x$$

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$$\begin{aligned}
y(x) &= \frac{1}{\frac{1}{x}} \int_1^x \left(\frac{1}{s}\right)(se^s)\, ds \quad + \quad \frac{e-1}{\frac{1}{x}} \\
&= x \int_1^x e^s ds \quad + \quad (e-1)\, x \\
&= x\left(e^x - e\right) + (e-1)\, x \\
&= xe^x - x
\end{aligned}$$

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Thus,

$$y(x) = xe^x - x$$

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This worked because if $H_1(x)$ and $H_2(y)$ are defined by (5) then we can derive (3) from (4) via implicit differentiation.

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A more general algebraic relationship between x and y can be expressed as

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I'll now derive a differential equation from (6) using implicit differentiation.

But first, we'll need a generalization of the chain rule.

Digression: The Chain Rule for Functions of 2 variables

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$$\frac{d}{dx}\left(f\left(y\left(x\right)\right)\right) = \frac{df}{dy}\frac{dy}{dx}$$

If s, t are functions of x and y, then

$$\begin{aligned} \frac{\partial}{\partial x}F\left(s\left(x,y\right),t\left(x,y\right)\right) &= \frac{\partial F}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial F}{\partial t}\frac{\partial t}{\partial x} \\ \frac{\partial}{\partial y}F\left(s\left(x,y\right),t\left(x,y\right)\right) &= \frac{\partial F}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial F}{\partial t}\frac{\partial t}{\partial y} \end{aligned}$$

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Now suppose we carry out implicit differentiation of (6), from our multi-variable chain rule we have

$$\frac{d}{dx}\left(\Phi\left(x, y\left(x\right)\right)\right) = \frac{d}{dx}\left(C\right) \quad \Rightarrow \quad \frac{\partial \Phi}{\partial x}\frac{dx}{dx} + \frac{\partial \Phi}{\partial y}\frac{dy}{dx} = 0 \qquad (7)$$

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or

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N.B., The term $\frac{\partial \Phi}{\partial x}$ and the factor $\frac{\partial \Phi}{\partial y}$ will, in general, depend on both x and y.

Exact Equations, Cont'd

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But there is also an additional hidden condition on the functions $M(x, y)$ and $N(x, y)$. Since

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial \Phi}{\partial y}\right) = \frac{\partial N}{\partial x}$$

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Thus, if (9) is to be derivable from an algebraic equation of the form $\Phi(x,y) = C$, we must have

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations, Cont'd

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Definition

A differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is called **exact**, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Theorem

Suppose

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{10}$$

is exact (i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$*).*

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with $\Phi(x,y)$ *determined (up to a constant) by the conditions*

$$\frac{\partial \Phi}{\partial x} = M(x,y) \qquad , \qquad \frac{\partial \Phi}{\partial y} = N(x,y) \tag{12}$$

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Of course, this immediately raises the question of exactly how $\Phi(x,y)$ is determined by (12).

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To answer this question, we just need the 2-variable version of the Fundamental Theorem of Calculus.

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$$\Phi(x, y) = \int M(x, y)\, \partial x \quad + \quad h_1(y)$$

where $h_1(y)$ is an arbitrary function depending only on y.
Similiarly, the most general solution of

$$\frac{\partial \Phi}{\partial y} = N(x, y)$$

is

$$\Phi(x, y) = \int N(x, y)\, \partial y \quad + \quad h_2(x)$$

Exact Equation Example

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Noting that

$$\frac{\partial M}{\partial y}=4xy=\frac{\partial N}{\partial x}$$

we confirm that the differential equation is exact.

Exact Equations Example, Cont'd

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The theorem about exact equations tells that (13) will have the same solutions as

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Exact Equations Example, Cont'd

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with $\Phi(x, y)$ determined by

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= M(x, y) = 2xy^2 + 1 \\ \frac{\partial \Phi}{\partial y} &= N(x, y) = 2x^2 y \end{aligned}$$

Exact Equations Example, Cont'd

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Applying our 2-variable Fundamental Theorem of Calculus

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= 2xy^2 + 1 \\ \Rightarrow \quad \Phi(x, y) &= \int \left(2xy^2 + 1\right) \partial x + h_1(y) \\ &= x^2 y^2 + x + h_1(y) \end{aligned}$$

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Then both equations say

$$\Phi(x, y) = x^2y^2 + x$$

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or

$$y(x) = \pm\sqrt{\frac{C - x}{x^2}} \tag{14}$$

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(14) will thus be the general solution to the exact equation (13).

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Summary: Solving Exact Equations, Cont'd

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3. $\Phi(x, y)$ is determined by calculating

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\Phi_1(x, y) &= \int M(x, y)\, \partial x + h_1(y) \\
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and then adjusting the arbitrary functions $h_1(y)$ and $h_2(x)$ so that $\Phi_1(x, y) = \Phi_2(x, y)$

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4. We then set $\Phi(x, y) = \Phi_1(x, y) \quad (= \Phi_2(x, y))$ and solve (if possible)

$$\Phi(x, y) = C \tag{17}$$

   for y as a function of x and C

Summary: Solving Exact Equations, Cont'd

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It sometimes happens that the algebraic equation

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ends up being a transcendental equation which can not be explicitly solved for y.

Summary: Solving Exact Equations, Cont'd

It sometimes happens that the algebraic equation

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ends up being a transcendental equation which can not be explicitly solved for y.
In such cases, one just stops at equation (17) and refers to equation (17) as the **implicit solution** of (15).

Exact Equation Example 2

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Consider

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For this differential equation, we have

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we have

$$\begin{aligned} \frac{\partial M}{\partial y} &= 2 \\ \frac{\partial N}{\partial x} &= 2 \end{aligned}$$

and so the equation **is exact**.

Exact Equations Example 2, Cont'd

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We now try to find the function $\Phi(x, y)$ so that solutions of (18) can be found by solving $\Phi(x, y) = C$

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$$
\begin{aligned}
\Phi_1(x, y) &= \int M(x, y)\,\partial x + h_1(y) = \int (x + 2y)\,\partial x + h_1(y) \\
&= \frac{1}{2}x^2 + 2xy + h_1(y) \\
\Phi_2(x, y) &= \int N(x, y)\,\partial y + h_2(x) = \int (2x + y)\,\partial y + h_2(x) \\
&= 2xy + \frac{1}{2}y^2 + h_2(x)
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To get $\Phi_1(x, y) = \Phi_2(x, y)$, we need to set

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To get $\Phi_1(x, y) = \Phi_2(x, y)$, we need to set

$$\begin{aligned}
h_1(y) &= \frac{1}{2}y^2 \\
h_2(x) &= \frac{1}{2}x^2
\end{aligned}$$

Exact Equations Example 2, Cont'd

Exact Equations Example 2, Cont'd

Thus, with these choices for $h_1(y)$ and $h_2(x)$

$$\Phi(x, y) = \frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2$$

Exact Equations Example 2, Cont'd

The last step is to set $\Phi(x, y)$ equal to a constant C and solve for y :

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$$\frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2 = C$$

$$\Rightarrow y = \frac{-2x \pm \sqrt{(2x)^2 - 4\left(\frac{1}{2}\right)\left(\frac{1}{2}x^2 - C\right)}}{2\left(\frac{1}{2}\right)}$$

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where I have applied the Quadratic Formula

$$ay^2 + by + c = 0 \quad \Rightarrow \quad y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$