Math 2233 - Lecture 7

Agenda

1. Review of Methods of Solving First Order ODEs
   - Easiest Case: $\frac{dy}{dx} = f(x)$
   - Separable Equations: $M(x) + N(y)\frac{dy}{dx} = 0$
   - Linear Equations: $\frac{dy}{dx} + p(x)y = g(x)$
   - Exact Equations: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Change of Variables

Solving 1st Order ODEs: the Easiest Case

Standard Form:

$$\frac{dy}{dx} = f(x) \qquad (1)$$

(1) $\Longrightarrow$ $y(x)$ must be an anti-derivative of $f(x)$.

Method: Integrate both sides and add in a arbitrary constant C to obtain the most general anti-derivative of $f(x)$ as a solution.

$$\frac{dy}{dx} = f(x) \quad \Longrightarrow \quad y(x) = \int f(x)\,dx \; + \; C$$

Solving 1st Order ODEs: Separable Equations

Standard Form:

$$M(x) + N(y)\frac{dy}{dx} = 0 \tag{2}$$

Separable ODEs are derivable from algebraic equations of the form

$$H_1(x) + H_2(y) = C \tag{3}$$

Method:

- Transform ODE into form (2) and identify the functions $M(x)$ and $N(y)$ correctly.
- Calculate

$$H_1(x) = \int M(x)\,dx \quad , \quad H_2(y) = \int N(y)\,dy$$

  and plug your results into the equation (3)
  This will yield the **implicit solution** of (3)
- If possible, solve (algebraically) the implicit solution to determine y as a function of x and C.

Solving 1st Order ODEs: Linear 1st Order ODEs

Standard Form:

$$\frac{dy}{dx} + p(x)y = g(x) \tag{4}$$

Method:

- Transform linear ODE into the standard form (4) to correctly identify the coefficient functions $p(x)$ and $g(x)$
- Compute the **integrating factor** $\mu(x)$

$$\mu(x) = \exp\left[\int p(x)\,dx\right] \tag{5}$$

- Compute the general solution of (4) as

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx \quad + \quad \frac{C}{\mu(x)} \tag{6}$$

Solving 1st Order ODEs: Exact Equations

Standard Form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{with} \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{7}$$

Exact Equations are ODEs that are derivable from algebraic equations of the form

$$\Phi\,(x,y) = C \tag{8}$$

Method:

- Verify that the equation is exact (i.e, that the ODE is of the form (7))
- Compute

$$\begin{aligned} \Phi_1(x,y) &= \int M(x,y)\,\partial x \;+\; c_1(y) \\ \Phi_2(x,y) &= \int N(x,y)\,\partial y \;+\; c_2(x) \end{aligned}$$

- Adjust the arbitrary functions $c_1(y)$ and $c_2(x)$ so that $\Phi_1(x,y) = \Phi_2(x,y) \equiv \Phi(x,y)$

Solving 1st Order ODEs: Exact Equations, Cont'd

- Insert the calculated $\Phi(x, y)$ into

$$\Phi(x, y) = C$$

  to obtain the **implicit solution** of (7)
- If possible, solve the implicit solution for y as a function of x and C.

Remark

Make sure your ODE is in the correct form **before** applying any of the proceding methods.

Examples:

(i) :

$$xe^y + y^2 \frac{dy}{dx} = 0$$

Divide both sides by e^y

$$\Rightarrow \quad x + e^{-y} y^2 \frac{dy}{dx} = 0$$

which is separable with $M(x) = x$ and $N(y) = e^{-y} y^2$

(ii)

$$x \frac{dy}{dx} + 2y + e^x = 0$$

Divide both sides by x to get

$$\frac{dy}{dx} + \frac{2}{x} y = -\frac{e^x}{x}$$

which is linear with $p(x) = \frac{2}{x}$ and $g(x) = -\frac{e^x}{x}$

(iii)

$$2x^2y^2 + x + 2x^3y\frac{dy}{dx} = 0$$

Not exact since

$$\frac{\partial}{\partial y}\left(2x^2y^2 + x\right) = 4x^2y \quad \neq \quad 6x^2y = \frac{\partial}{\partial x}\left(2x^3y\right)$$

Divide ODE by x,

$$2xy^2 + 1 + 2x^2y\frac{dy}{dx} = 0$$

Noting

$$\frac{\partial}{\partial y}\left(2xy^2\right) = 4xy = \frac{\partial}{\partial x}\left(2x^2y\right)$$

we see that the new equation is exact with

$$M(x,y) = 2xy^2 + 1 \quad , \quad N(x,y) = 2x^2y$$

Change of Variables

In each of the above examples, we used simple algebraic operations to get the ODEs to a solvable standard form.

There is one more thing we could try to get an ODE into a solvable standard form: make a change of variables.

Change of Variables: Example

Consider

$$y' = (x+y)^2 \tag{9}$$

This ODE is Not Separable, Not Linear, and Not Exact.

Set $z = x + y$
Then

$$y = z - x$$

and so

$$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

Separately substitute for y and $\frac{dy}{dz}$ in (9)

$$\frac{dz}{dx} - 1 = z^2$$

Change of Variables Example, Cont'd

or

$$1 - \frac{1}{z^2 + 1} \frac{dz}{dx} = 0 \qquad (10)$$

Equation (10) is Separable with

$$M(x) = 1 \quad , \quad N(z) = \frac{-1}{z^2 + 1}$$

So the solutions of (10) coincide with solutions of

$$C = \int M(x)\, dx + \int N(z)\, dz = \int 1 dx \; - \; \int \frac{1}{1 + z^2} dz$$

or

$$x - \tan^{-1}(z) = C$$

So the function $z(x)$ must be

$$z = \tan(x - C) \qquad (11)$$

We now need to back substitute $z = x + y$ to get the solution to the original differential equation

Change of Variables Example, Cont'd

Replacing z in (11) by $x + y$ we find

$$(x + y) = \tan(x - C)$$

So

$$y = \tan(x - C) - x$$

This is the general solution of the original ODE.

Remarks

1. The key thing to remember is that both y and $\frac{dy}{dx}$ have to be separately substituted for.
2. Use the reverse relationship $y = f(x, z)$ to calculate $\frac{dy}{dx}$ in terms of $x, z, \frac{dz}{dx}$
3. Not always easy to guess what substitution might be helpful.

Change of Variables for Equations of Homogeneous Type

A 1st order ODE of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \tag{12}$$

is said to be of **homogeneous type**.
In this situation, the substitution

$$z = \frac{y}{x}$$

always leads to a Separable ODE for $z(x)$.

Substitution for ODEs of the form $\frac{dy}{dx} = F(\frac{y}{x})$

Let

$$z = \frac{y}{x} \quad \Rightarrow \quad y = zx \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z$$

And so substituting for y and $\frac{dy}{dx}$, in (12) yields

$$x\frac{dz}{dx} + z = F(z)$$

Multiplying by $\frac{1}{x(F(z)-z)}$ yields

$$\frac{1}{x} - \frac{1}{F(z) - z}\frac{dz}{dx} = 0$$

which is separable with

$$\begin{aligned} M(x) &= \frac{1}{x} \\ N(z) &= -\frac{1}{F(z) - z} \end{aligned}$$

Example: Substitution for ODE of form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$

$$x^2 \frac{dy}{dx} = xy + y^2$$

Divide by x^2

$$\frac{dy}{dx} = \frac{xy + y^2}{x^2} = \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2$$

Now introduce a change of variables

$$\begin{aligned} z &= \frac{y}{x} \\ &\Rightarrow \quad y = zx \\ &\Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z \end{aligned}$$

Substitute these expressions for y and $\frac{dy}{dx}$ back into (10)

$$x\frac{dz}{dx} + z = z + z^2$$

Example, Cont'd

or, after dividing both sides by xz^2 and simplying,

$$\frac{1}{x} - \frac{1}{z^2}\frac{dz}{dx} = 0$$

which is Separable with

$$M(x) = \frac{1}{x} \quad , \quad N(z) = -\frac{1}{z^2}$$

$$\begin{aligned} C &= \int M(x)\,dx + \int N(z)\,dz \\ &= \int \frac{1}{x}dx \quad - \quad \int \frac{1}{z^2}dz \\ &= \ln|x| + \frac{1}{z} \end{aligned}$$

and so

$$z(x) = \frac{1}{C - \ln|x|}$$

Finally, we rewrite in terms of original variable y:

$$z = \frac{y}{x} \quad \Rightarrow \quad \frac{y}{x} = \frac{1}{C - \ln|x|} \quad \Rightarrow \quad y(x) = \frac{x}{C - \ln|x|}$$

Example: HW Problem 2.4.21

Solve

$$\begin{aligned} \frac{1}{x} + 2y^2x + \left(2yx^2 - \cos(y)\right)\frac{dy}{dx} &= 0 \\ y(1) &= \pi \end{aligned}$$

Let's first find the general solution to the ODE. With

$$\begin{aligned} M(x,y) &= \frac{1}{x} + 2y^2x \\ N(x,y) &= 2yx^2 - \cos(y) \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{x} + 2y^2x\right) = 0 + 4xy \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2yx^2 - \cos(y)\right) = 4xy - 0 \end{aligned}$$

Example: HW Problem 2.4.21, Cont'd

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the ODE is **exact**. This means that the ODE has the same solution as an algebraic equation of the form

$$\Phi(x, y) = C$$

with $\Phi(x, y)$ determined by

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= M(x, y) = \frac{1}{x} + 2y^2 x \\ \frac{\partial \Phi}{\partial y} &= N(x, y) = 2yx^2 - \cos(y) \end{aligned}$$

Example: HW Problem 2.4.21, Cont'd

To recover $\Phi(x, y)$ from these conditions, we simply take anti-partial derivatives by integrating both sides (and adding in some arbitrary functions of the other variables). Thus,

$$\begin{aligned}\Phi(x, y) &= \int M(x, y)\,\partial x + c_1(y) \\ &= \int \left(\frac{1}{x} + 2y^2 x\right)\partial x + c_1(y) \\ &= \ln|x| + y^2 x^2 + c_1(y)\end{aligned}$$

and

$$\begin{aligned}\Phi(x, y) &= \int N(x, y)\,\partial y + c_2(x) \\ &= \int \left(2yx^2 - \cos(y)\right)\partial y + c_2(x) \\ &= y^2 x^2 - \sin(y) + c_2(x)\end{aligned}$$

Example: HW Problem 2.4.21, Cont'd

These two expressions for $\Phi(x, y)$ agree only if

$$\begin{aligned} c_1(y) &= -\sin(y) \\ c_2(x) &= \ln|x| \end{aligned}$$

Thus, with these choices,

$$\Phi(x, y) = x^2 y^2 + \ln|x| - \sin(y)$$

and the implicit solution $\Phi(x, y) = C$ to the orginal differential equation will be

$$x^2 y^2 + \ln|x| - \sin(y) = C \qquad (*)$$

This is actually a transcendental equation in y, so we won't be able to solve it explicitly to get y as a function of x.

Example: HW Problem 2.4.21, Cont'd

However, we can still determine the correct value for the constant C by imposing the initial condition $y(1) = \pi$. Substituting $x = 1$ and $y = \pi$ into (*) yields

$$(1)^2 (\pi)^2 + \ln|1| - \sin(\pi) = C \quad \Rightarrow \quad C = \pi^2$$

Thus, the implicit solution to the initial value problem will be

$$x^2 y^2 + \ln|x| - \sin(y) = \pi^2$$

Example: HW Problem 2.6.24

$$\frac{dy}{dx} = \frac{y\left(\ln|y| - \ln|x| + 1\right)}{x} \qquad (*)$$

This is to be an ODE of homogeneous type; i.e., an ODE of the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \qquad (**)$$

So we want to get the right hand side of (*) into the form of a function of the ratio of y to x. We have the following identity for the natural log function:

$$\ln|y| - \ln|x| = \ln\left|\frac{y}{x}\right|$$

Thus, the right hand side of (*) can be written

$$\frac{y}{x}\left(\ln\left|\frac{y}{x}\right| + 1\right)$$

Hence, by defining

$$F(u) = u\left(\ln|u| + 1\right)$$

we get (*) in the form (**).

Example: HW Problem 2.6.24, Cont'd

Good. Now we can try the change of variable

$$\begin{aligned} z &= \frac{y}{x} \\ &\Rightarrow \quad y = zx \\ &\Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z \end{aligned}$$

In (*) we now substitute $\left(x\frac{dz}{dx} + za\right)$ for $\frac{dy}{dx}$ and z for $\frac{y}{x}$, to get

$$x\frac{dz}{dx} + z = z\left(\ln|z| + 1\right) = z\ln|z| + z$$

Cancelling the z term that appears on both sides we get

$$x\frac{dz}{dx} = z\ln|z|$$

or, after dividing by $xz\ln|z|$ are rearranging terms:

$$\frac{1}{x} - \frac{1}{z\ln|z|}\frac{dz}{dx} = 0 \qquad (***)$$

This is a Separable ODE for $z(x)$.

Example: HW Problem 2.6.24, Cont'd

Next, we solve the separable ODE for $z(x)$. Thus, we calculate

$$H_1(x) = \int M(x)\,dx = \int \frac{1}{x}dx = \ln|x|$$

and

$$H_2(z) = \int N(z)\,dz = -\int \frac{1}{z\ln|z|}dz$$

This last integral can be carried out using the substitution

$$u = \ln|z| \quad \Rightarrow \quad du = \frac{1}{z}dz$$

and so

$$-\int \frac{1}{\ln|z|}\left(\frac{1}{z}dz\right) = -\int \frac{1}{u}du = -\ln|u| = -\ln(\ln|z|)$$

and so

$$H_2(z) = -\ln(\ln|z|)$$

Example: HW Problem 2.6.24, Cont'd

The solution to the separable equation (***) is thus given implicitly by

$$H_1(x) + H_2(z) = C$$

or

$$\ln|x| - \ln(\ln|z|) = C$$

Now we have to convert back to our original unknown function $y(x)$.
Since, by our original change of variables,

$$z = \frac{y}{x}$$

we have

$$\ln|x| - \ln\left(\ln\left|\frac{y}{x}\right|\right) = C$$

or

$$\ln\left(\ln\left|\frac{y}{x}\right|\right) = C - \ln|x|$$

Example: HW Problem 2.6.24, Cont'd

or, after exponentiating both sides,

$$\ln\left|\frac{y}{x}\right| = \exp\left(C - \ln|x|\right)$$

exponentiating again,

$$\frac{y}{x} = \exp\left[\exp\left(C - \ln|x|\right)\right]$$

or

$$y = x\exp\left[\exp\left(C - \ln|x|\right)\right]$$

Now

$$\exp\left(C - \ln|x|\right) = e^{C}\exp\left(\ln|x|\right) = e^{C}x$$

Thus,

$$y\left(x\right) = x\exp\left(e^{C}x\right)$$