Math 2233 - Lecture 7

Agenda

1. Review of Methods of Solving First Order ODEs
   - Easiest Case: $\frac{dy}{dx} = f(x)$
   - Separable Equations: $M(x) + N(y)\frac{dy}{dx} = 0$
   - Linear Equations: $\frac{dy}{dx} + p(x)y = g(x)$
   - Exact Equations: $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ with $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Change of Variables

Solving 1st Order ODEs: the Easiest Case

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$$\frac{dy}{dx} = f(x) \quad \Longrightarrow \quad y(x) = \int f(x)\,dx \; + \; C$$

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- If possible, solve (algebraically) the implicit solution to determine y as a function of x and C.

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Method:

- Transform linear ODE into the standard form (4) to correctly identify the coefficient functions $p(x)$ and $g(x)$
- Compute the **integrating factor** $\mu(x)$

$$\mu(x) = \exp\left[\int p(x)\,dx\right] \tag{5}$$

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- Compute the general solution of (4) as

$$y(x) = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx \quad + \quad \frac{C}{\mu(x)} \tag{6}$$

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- Compute

$$\begin{aligned} \Phi_1(x,y) &= \int M(x,y)\,\partial x \; + \; c_1(y) \\ \Phi_2(x,y) &= \int N(x,y)\,\partial y \; + \; c_2(x) \end{aligned}$$

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\end{aligned}$$

- Adjust the arbitrary functions $c_1(y)$ and $c_2(x)$ so that $\Phi_1(x,y) = \Phi_2(x,y) \equiv \Phi(x,y)$

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- Insert the calculated $\Phi(x, y)$ into

$$\Phi(x, y) = C$$

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- Insert the calculated $\Phi(x, y)$ into

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- If possible, solve the implicit solution for y as a function of x and C.

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which is separable with $M(x) = x$ and $N(y) = e^{-y} y^2$

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(ii)

$$x \frac{dy}{dx} + 2y + e^x = 0$$

Divide both sides by x to get

$$\frac{dy}{dx} + \frac{2}{x} y = -\frac{e^x}{x}$$

which is linear with $p(x) = \frac{2}{x}$ and $g(x) = -\frac{e^x}{x}$

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we see that the new equation is exact with

$$M(x, y) = 2xy^2 + 1 \quad , \quad N(x, y) = 2x^2y$$

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There is one more thing we could try to get an ODE into a solvable standard form: make a change of variables.

Change of Variables: Example

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Consider

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Separately substitute for y and $\frac{dy}{dz}$ in (9)

$$\frac{dz}{dx} - 1 = z^2$$

Change of Variables Example, Cont'd

or

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or

$$1 - \frac{1}{z^2 + 1} \frac{dz}{dx} = 0 \tag{10}$$

Change of Variables Example, Cont'd

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$$1 - \frac{1}{z^2+1}\frac{dz}{dx} = 0 \tag{10}$$

Equation (10) is Separable with

$$M(x) = 1 \quad , \quad N(z) = \frac{-1}{z^2+1}$$

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Equation (10) is Separable with

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So the solutions of (10) coincide with solutions of

$$C = \int M(x)\,dx + \int N(z)\,dz = \int 1dx \; - \; \int \frac{1}{1+z^2}dz$$

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So the function $z(x)$ must be

$$z = \tan(x - C) \tag{11}$$

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So the function $z(x)$ must be

$$z = \tan(x - C) \qquad (11)$$

We now need to back substitute $z = x + y$ to get the solution to the original differential equation

Change of Variables Example, Cont'd

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Replacing z in (11) by $x + y$ we find

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So

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This is the general solution of the original ODE.

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2. Use the reverse relationship $y = f(x, z)$ to calculate $\frac{dy}{dx}$ in terms of $x, z, \frac{dz}{dx}$
3. Not always easy to guess what substitution might be helpful.

Change of Variables for Equations of Homogeneous Type

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In this situation, the substitution

$$z = \frac{y}{x}$$

always leads to a Separable ODE for $z(x)$.

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which is separable with

$$M(x) = \frac{1}{x}$$

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Now introduce a change of variables

$$\begin{aligned} z &= \frac{y}{x} \\ &\Rightarrow \quad y = zx \\ &\Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z \end{aligned}$$

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Substitute these expressions for y and $\frac{dy}{dx}$ back into (10)

$$x\frac{dz}{dx} + z = z + z^2$$

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which is Separable with

$$M(x) = \frac{1}{x} \quad , \quad N(z) = -\frac{1}{z^2}$$

$$\begin{aligned} C &= \int M(x)\,dx + \int N(z)\,dz \\ &= \int \frac{1}{x}dx \quad - \quad \int \frac{1}{z^2}dz \\ &= \ln|x| + \frac{1}{z} \end{aligned}$$

and so

$$z(x) = \frac{1}{C - \ln|x|}$$

and so

$$z\left(x\right)=\frac{1}{C-\ln\left|x\right|}$$

Finally, we rewrite in terms of original variable y:

and so

$$z(x) = \frac{1}{C - \ln|x|}$$

Finally, we rewrite in terms of original variable y:

$$z = \frac{y}{x} \quad \Rightarrow \quad \frac{y}{x} = \frac{1}{C - \ln|x|} \quad \Rightarrow \quad y(x) = \frac{x}{C - \ln|x|}$$

Example: HW Problem 2.4.21

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Solve

$$\begin{aligned}\frac{1}{x} + 2y^2x + \left(2yx^2 - \cos(y)\right)\frac{dy}{dx} &= 0 \\ y(1) &= \pi\end{aligned}$$

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we have

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{1}{x} + 2y^2 x\right) = 0 + 4xy \\ \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2yx^2 - \cos(y)\right) = 4xy - 0 \end{aligned}$$

Example: HW Problem 2.4.21, Cont'd

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the ODE is **exact**.

Example: HW Problem 2.4.21, Cont'd

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the ODE is **exact**. This means that the ODE has the same solution as an algebraic equation of the form

$$\Phi(x, y) = C$$

Example: HW Problem 2.4.21, Cont'd

Thus, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and so the ODE is **exact**. This means that the ODE has the same solution as an algebraic equation of the form

$$\Phi(x, y) = C$$

with $\Phi(x, y)$ determined by

$$\begin{aligned} \frac{\partial \Phi}{\partial x} &= M(x, y) = \frac{1}{x} + 2y^2 x \\ \frac{\partial \Phi}{\partial y} &= N(x, y) = 2yx^2 - \cos(y) \end{aligned}$$

Example: HW Problem 2.4.21, Cont'd

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To recover $\Phi(x, y)$ from these conditions, we simply take anti-partial derivatives by integrating both sides (and adding in some arbitrary functions of the other variables).

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$$\begin{aligned}
\Phi(x, y) &= \int M(x, y)\,\partial x + c_1(y) \\
&= \int \left(\frac{1}{x} + 2y^2 x\right) \partial x + c_1(y) \\
&= \ln|x| + y^2 x^2 + c_1(y)
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&= y^2 x^2 - \sin(y) + c_2(x)
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Example: HW Problem 2.4.21, Cont'd

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These two expressions for $\Phi(x, y)$ agree only if

$$\begin{aligned} c_1(y) &= -\sin(y) \\ c_2(x) &= \ln|x| \end{aligned}$$

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Thus, with these choices,

$$\Phi(x, y) = x^2 y^2 + \ln|x| - \sin(y)$$

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and the implicit solution $\Phi(x, y) = C$ to the orginal differential equation will be

$$x^2y^2 + \ln|x| - \sin(y) = C \qquad (*)$$

This is actually a transcendental equation in y, so we won't be able to solve it explicitly to get y as a function of x.

Example: HW Problem 2.4.21, Cont'd

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However, we can still determine the correct value for the constant C by imposing the initial condition $y(1) = \pi$.

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$$(1)^2 (\pi)^2 + \ln|1| - \sin(\pi) = C \quad \Rightarrow \quad C = \pi^2$$

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$$(1)^2 (\pi)^2 + \ln|1| - \sin(\pi) = C \quad \Rightarrow \quad C = \pi^2$$

Thus, the implicit solution to the initial value problem will be

$$x^2 y^2 + \ln|x| - \sin(y) = \pi^2$$

Example: HW Problem 2.6.24

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$$\frac{dy}{dx} = \frac{y\left(\ln|y| - \ln|x| + 1\right)}{x} \qquad (*)$$

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This is to be an ODE of homogeneous type; i.e., an ODE of the form

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Hence, by defining

$$F(u) = u\left(\ln|u| + 1\right)$$

Example: HW Problem 2.6.24, Cont'd

Good. Now we can try the change of variable

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$$\begin{aligned} z &= \frac{y}{x} \\ &\Rightarrow \quad y = zx \\ &\Rightarrow \quad \frac{dy}{dx} = x\frac{dz}{dx} + z \end{aligned}$$

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In (*) we now substitute $\left(x\frac{dz}{dx} + za\right)$ for $\frac{dy}{dx}$ and z for $\frac{y}{x}$, to get

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Cancelling the z term that appears on both sides we get

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or, after dividing by $xz\ln|z|$ are rearranging terms:

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This is a Separable ODE for $z(x)$.

Example: HW Problem 2.6.24, Cont'd

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Next, we solve the separable ODE for $z(x)$. Thus, we calculate

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$$H_1(x) = \int M(x)\, dx = \int \frac{1}{x} dx = \ln|x|$$

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$$H_1(x) = \int M(x)\,dx = \int \frac{1}{x}dx = \ln|x|$$

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Example: HW Problem 2.6.24, Cont'd

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The solution to the separable equation (***) is thus given implicitly by

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$$\ln|x| - \ln(\ln|z|) = C$$

Example: HW Problem 2.6.24, Cont'd

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Now we have to convert back to our original unknown function $y(x)$.

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$$\ln|x| - \ln\left(\ln\left|\frac{y}{x}\right|\right) = C$$

or

$$\ln\left(\ln\left|\frac{y}{x}\right|\right) = C - \ln|x|$$

Example: HW Problem 2.6.24, Cont'd

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or, after exponentiating both sides,

Example: HW Problem 2.6.24, Cont'd

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$$\ln\left|\frac{y}{x}\right| = \exp\left(C - \ln|x|\right)$$

Example: HW Problem 2.6.24, Cont'd

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exponentiating again,

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Example: HW Problem 2.6.24, Cont'd

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$$\exp\left(C - \ln|x|\right) = e^{C}\exp\left(\ln|x|\right) = e^{C}x$$

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$$y\left(x\right) = x\exp\left(e^{C}x\right)$$