Math 2233 - Lecture 9

Agenda:

1. 2nd Order Linear ODEs, Standard Form
2. The Existence and Uniqueness Theorem
3. Differential Operator Notation
4. Homogeneous vs. Nonhomogeneous Linear ODEs
5. Two Fundamental Theorems for Homogeneous 2nd Order Linear ODEs
   - The Superposition Principle
   - The Completeness Theorem

Standard Form of a 2nd Order Linear ODE

A second order linear differential equation is a differential equation of the form

$$A(x)y'' + B(x)y' + C(x)y = D(x) \quad . \tag{1}$$

(Here A, B, C and D are prescribed functions of x.)
As in the case of first order linear equations, in any interval where $A(x) \neq 0$, we can replace such an equation by an equivalent one in the **standard form**:

$$y'' + p(x)y' + q(x)y = g(x) \tag{2}$$

where

$$\begin{aligned} p(x) &= \frac{B(x)}{A(x)} \\ q(x) &= \frac{C(x)}{A(x)} \\ g(x) &= \frac{D(x)}{A(x)} \end{aligned}$$

The Simplest Example

Consider the ODE

$$\frac{d^2x}{dt^2} = 0 \tag{3}$$

We can solve this using just the Fundamental Theorem of Calculus in two steps.

Let $v \equiv \frac{dx}{dt}$. Substituting into (3),

$$\frac{dv}{dt} = 0 \quad \Rightarrow \quad v(t) = v_0 \quad , \quad v_0 \text{ a constant}$$

But now $v_0 = v(t) = \frac{dx}{dt}$ implies

$$\frac{dx}{dt} = v_0 \quad \Rightarrow \quad x(t) = \int v_0 dt + x_0 = v_0 t + x_0$$

Thus, the general solution of (4) is

$$x(t) = v_0 t + x_0$$

with v_0 and x_0 constants.

Physical Intepretation of the Constants v_0 and x_0

So the general solution of

$$\frac{d^2x}{dt^2} = 0$$

is

$$x(t) = v_0 t + x_0$$

Note that we have

$$\begin{aligned} x(0) &= 0 + x_0 = x0 \\ \frac{dx}{dt}(0) &= (v_0 + 0)\big|_{t=0} = v_0 \end{aligned}$$

Physical Interpretation: If the acceleration of an object is 0, then its trajectory x(t) is completely determined by its initial position x_0 and its initial velocity v_0.
This is, effectively, Newton's 1st Law of Motion.

Moral of the Simple Example

- The general solution of a 2nd Order Linear ODE will involve 2 arbitrary constants. (Below I'll refer to them as c_1 and c_2).
- To select a unique solution, two additional conditions will be needed. (As we'll need two equations to solve for two unknowns c_1 and c_2)

In the 1st Order case, we used an initial condition

$$y(x_0) = y_0$$

to fix a unique solution.
For 2nd Order ODEs, an additional initial condition is needed. The $\frac{d^2x}{dt^2} = 0$ example suggests using

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y'_0 \end{aligned}$$

Existence and Uniqueness Theorem

Here is the precise statement about the existence and uniqueness of solutions to homogeneous 2nd order linear ODEs.

Theorem

If the functions p, q and g are continuous on an open interval $I \subset \mathbb{R}$ containing the point x_o, then in some interval about x_o, then there exists a unique solution to the differential equation

$$y'' + p(x)y' + q(x)y = g(x)$$

satisfying the initial conditions of the form

$$\begin{array}{rcl} y(x_o) & = & y_o \\ y'(x_o) & = & y'_o \end{array} \quad .$$

Summary and Remarks

- For any reasonable choice of functions $p(x)$, $q(x)$, and $g(x)$, we will always have solutions to

$$y'' + p(x)y' + q(x)y = g(x) \tag{2}$$

- The general solution of (2) will always involve two arbitary constants
- There is only one solution if initial conditions

$$\begin{aligned} y(x_o) &= y_o \\ y'(x_o) &= y'_o \end{aligned}$$

are imposed on the general solution.

- The E&U theorem does not address the issue of how to find solutions of a second order linear ODE.
- In fact, finding solutions to ODEs of the form (2) is going to be the main problem for the rest of the course.

Differential Operator Notation

Consider the general second order linear differential equation

$$\phi'' + p(x)\phi' + q(x)\phi = g(x) \quad .$$

For notational ease, we shall often write differential equations like this as

$$L[\phi] = g(x)$$

where L is the **linear differential operator**

$$L = \frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x) \quad .$$

This will mean L acts on a function $\phi(x)$ by

$$\begin{aligned} L[\phi] &= \left(\tfrac{d^2}{dx^2} + p(x)\tfrac{d}{dx} + q(x)\right)\phi \\ &= \tfrac{d^2\phi}{dx^2} + p(x)\tfrac{d\phi}{dx} + q(x)\phi \quad . \end{aligned}$$

Homogeneous vs. Non-homogeneous Linear Differential Equations

Rather than solve ODEs of the form

$$y'' + p(x)y' + q(x)y = g(x) \tag{2}$$

directly, we will first study some simpler subcases.
In fact, in what follows, it will be important first to distinguish between the cases when the function $g(x)$ on the right hand side of (2) is zero or non-zero.

Definition

A second order linear ODE is said to be **homogeneous** if when written in the form (2), the function $g(x)$ on the right is identically 0. So a homogeneous 2nd order linear ODE has the form

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

An ODE of the form (2) with $g(x) \neq 0$) is said to be **nonhomogeneous**.

The Superposition Principle

Our first results will be about solutions of homogeneous 2nd order linear ODEs.

Theorem (The Superposition Principle)

Suppose $y_1(x)$ and $y_2(x)$ are two solutions of

$$L[y] = \frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

and c_1 and c_2 are constants. Then, any function of the form

$$y = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution to the differential equation.

Proof of the Superposition Principle

Given $L[y_1] = 0$ and $L[y_2] = 0$

$$
\begin{aligned}
L[c_1y_1 + c_2y_2] &= \frac{d^2}{dx^2}(c_1y_1 + c_2y_2) \\
&\quad + p(x)\frac{d}{dx}(c_1y_1 + c_2y_2) \\
&\quad + q(x)(c_1y_1 + c_2y_2) \\
&= c_1\left(\frac{d^2y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x)y_1\right) \\
&\quad + c_2\left(\frac{d^2y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x)y_2\right) \\
&= c_1L[y_1] + c_2L[y_2] \\
&= c_1 \cdot 0 + c_2 \cdot 0 \\
&= 0
\end{aligned}
$$

$\square$

Remarks

- The fact that a linear combination of solutions of a **linear, homogeneous differential equation** is also a solution is extremely important.
- For those of you who have had a little linear algebra, the Superposition Principle says that the solution set of $L[y] = 0$ is a vector space. (A set closed under scalar multiplication and vector addition)
- As such, a lot of the results of Math 3013 Linear Algebra will find direct application in the study of linear differential equations.
- However, since Math 3013 is not a prerequisite for Math 2233, I will only make passing remarks about the connections with Linear Algebra.

Example 1

Consider

$$y'' + y = 0$$

I claim both $y_1(x) = \cos(x)$ and $y_2 = \sin(x)$ are solutions. Indeed,

$$\begin{aligned}
\frac{d^2}{dx^2}\cos(x) + \cos(x) &= \frac{d}{dx}(-\sin(x)) + \cos(x) \\
&= -\cos(x) + \cos(x) \\
&= 0 \\
\frac{d^2}{dx^2}\sin(x) + \sin(x) &= \frac{d}{dx}(\cos(x)) + \sin(x) \\
&= -\sin(x) + \sin(x) \\
&= 0
\end{aligned}$$

Example 1, Cont'd

The Superposition Principle then tells us that any function of the form $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is also a solution of

$$y'' + y = 0$$

Indeed

$$\begin{aligned}
\frac{d^2y}{dx^2} + y &= \frac{d^2}{dx^2}\left(c_1 \cos(x) + c_2 \sin(x)\right) + \left(c_1 \cos(x) + c_2 \sin(x)\right) \\
&= \frac{d}{dx}\left(-c_1 \sin(x) + c_2 \cos(x)\right) + c_1 \cos(x) + c_2 \sin(x) \\
&= -c_1 \cos(x) - c_2 \sin(x) + c_1 \cos(x) + c_2 \sin(x) \\
&= 0
\end{aligned}$$

Pseudo-Counter-Examples

Consider

$$y'' + 2y'y = 0 \tag{5}$$

(Note this is a non-linear ODE, so the Superposition Principle is not applicable.) I claim both

$$y_1(x) = 1 \quad , \quad y_2(x) = x^{-1}$$

are solutions of (5). Indeed

$$\begin{aligned} y_1'' + 2y_1'y_1 &= 0 + 2(0)(1) = 0 \\ y_2'' + (y_2)^2 &= +2x^{-3} + 2\left(-x^{-2}\right)\left(x^{-1}\right) = 0 \end{aligned}$$

Now consider the following linear combination of y_1 and y_2

$$y_3 = y_1 + y_2 = 1 + x^{-1}$$

We have

$$y_3'' + 2y_3'y_2 = -2x^{-3} + 2\left(-x^{-2}\right)\left(1 + x^{-1}\right) = -2x^{-2} \neq 0$$

and so y_3 is not a solution of (5).

Pseudo-Counter-Examples, Cont'd

Conclusion: The Superposition Principle does not hold for nonlinear ODEs.

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Nor does it hold for **nonhomogeneous** 2nd Order Linear ODEs. If y_1 and y_2 are two solutions of

$$y'' + p(x)y' + q(x)y = g(x) \tag{2}$$

and we set $y_3 = y_1 + y_2$ then

$$\begin{aligned} y_3'' + p(x)\,y_3' + q(x)\,y_3 &= \left(y_1'' + y_2''\right) + p(x)\left(y_1' + y_2'\right) \\ &\quad + q(x)\left(y_1 + y_2\right) \\ &= \left(y_1'' + p(x)\,y_1' + q(x)\,y_1\right) \\ &\quad + \left(y_2'' + p(x)\,y_2' + q(x)\,y_2\right) \\ &= g(x) + g(x) \\ &= 2g(x) \\ &\neq g(x) \end{aligned}$$

Remarks

Once we have two solutions $y_1(x)$ and $y_2(x)$ of a second order linear homogeneous differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{6}$$

the Superposition Principle allows us to construct an infinite set of distinct solutions by setting

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{7}$$

and letting the constants c_1 and c_2 run through the real numbers $\mathbb{R}$.

The following question then arises

Are all the solutions of (6) expressable in form (7) for some choice of c_1 and c_2?

This will not always be the case; but when **every** solution of (6) can be written as (7), we shall say that the two solutions y_1 and y_2 form a **fundamental set** of solutions if every solution of (6) can be expressed in the form (7).

The Completeness Theorem

The following theorem provides a simple way to check if a pair of solutions is a fundamental set of solutions.

Theorem

Suppose y_1 and y_2 are two solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{6}$$

that satisfy

$$y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0 \tag{8}$$

then **every** *solution of (6) can be written in the form*

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

This theorem reduces to the problem of solving (6) completely, to the task of finding two solutions, y_1 and y_2, for which (8) holds.

Proof of the Completeness Theorem

Let y_1 and y_2 be two given solutions on an interval I and let Y be any other solution on I. Choose a point $x_o \in I$. From our the Existence and Uniqueness Theorem, we know that there is only one solution $y(x)$ of (6) such that

$$\begin{aligned} y(x_o) &= Y_0 \equiv Y(x_0) \\ y'(x_o) &= Y_0' \equiv Y'(x_0) \end{aligned} \tag{9}$$

namely, $Y(x)$.

Therefore if we can show that we can find numbers c_1 and c_2 such that the

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

(which is guaranteed to be a solution by the Superposition Principle) satisfies the initial conditions (9), then we must have

$$Y(x) = c_1 y_1(x) + c_2 y_2(x).$$

Proof of the Completeness Theorem, Cont'd

We now seek to find constants c_1 and c_2 so that these initial conditions can be matched. We thus set

$$\begin{aligned} c_1 y_1(x_o) + c_2 y_2(x_o) &= Y_o \\ c_1 y_1'(x_o) + c_2 y_2'(x_o) &= Y_o' \quad . \end{aligned} \tag{10}$$

This is just a series of two equations with two unknowns. Solving the first equation for c_1 yields

$$c_1 = \frac{Y_o - c_2 y_2(x_o)}{y_1(x_o)} \quad . \tag{11}$$

Plugging this into the second equation yields

$$\frac{Y_o - c_2 y_2(x_o)}{y_1(x_o)} y_1'(x_o) + c_2 y_2'(x_o) = Y_o'$$

or

$$Y_o y_1'(x_o) - c_2 y_2(x_o) y_1'(x_o) + c_2 y_1(x_o) y_2'(x_o) = y_1(x_o) Y_o'$$

Proof of the Completeness Theorem, Cont'd

Solving this last equation for c_2 yields

$$c_2 = \frac{y_1(x_o)Y_o' - y_1'(x_o)Y_o}{y_1(x_o)y_2'(x_o) - y_1'(x_o)y_2(x_o)} \tag{12}$$

Plugging this expression for c_2 into (11) yields

$$c_1 = \frac{Y_o y_2'(x_o) - y_2(x_o)Y_o'}{y_1(x_o)y_2'(x_o) - y_1'(x_o)y_2(x_o)} \quad . \tag{13}$$

So long as the denominators on the RHS of (12) and (13) are non-zero, we can thus find constants c_1 and c_2 such that an arbitrary solution $Y(x)$ and the solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$ satisfy the same initial conditions And so by Uniqueness part of the E&U Theorem

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

□

The Wronskian

Definition

The expression

$$W(y_1, y_2) = y_1(x_o)y_2'(x_o) - y_1'(x_o)y_2(x_o)$$

is called the **Wronskian** of y_1 and y_2 and the condition

$$W(y_1, y_2) \neq 0$$

is called the **Wronskian Condition**.

Two solutions $y_1(x)$ and $y_2(x)$ satisfying the Wronksian Condition form a fundamental set of solution. Sometimes, we'll call two solutions satisfying the Wronkskian Condition **independent solutions**.

Upshot: Solving 2nd Order, Linear, Homogeneous ODEs

To find **all** of the solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{4}$$

it suffices to

1. Find, or even guess, two functions $y_1(x)$ and $y_2(x)$ that satisfy the differential equation (4)
2. Check that y_1 and y_2 also satisfy the Wronskian Condition

$$0 \neq W[y_1, y_2]$$

3. One can then write down the general solution to (4) as

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{14}$$

Upshot Cont'd: Solving 2nd Order Initial Value Problems

To solve an initial value problem of the form

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(x_0) &= y_0 \\ y'(x_0) &= y_0' \end{aligned}$$

1. One first finds the general solution of the ODE (as outlined on previous slide)
2. One then uses the initial conditions to deduce two equations for the constants c_1 and c_2 appearing in the general solution.
3. Solving these equations for constants c_1 and c_2 identifies the particular solution that satisfies the initial condition, and thus solves the initial value problem.

Example

Find the unique solution to

$$\frac{d^2y}{dx^2} + y = 0 \tag{15}$$

satisfying

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

We saw in an earlier example, that both

$$y_1(x) = \cos(x) \quad \text{and} \quad y_2(x) = \sin(x)$$

satisfy the differential equation.
Let us check that their Wronskian does not vanish:

$$\begin{aligned} W(y_1, y_2) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= \cos(x)\,(\cos(x)) - (-\sin(x))\sin(x) \\ &= 1 \\ &\neq 0 \quad . \end{aligned}$$

Example, Cont'd

Good, since $W[y_1, y_2] \neq 0$, y_1 and y_2 will constitute a fundamental set of solutions for (15)
The general solution of (15) is thus

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = c_1 \cos(x) + c_2 \sin(x)$$

We now impose the initial conditions on the general solution:

$$\begin{aligned} 1 &= y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1 (1) + c_2 (0) = c_1 \\ 2 &= y'(0) = -c_1 \sin(0) + c_2 \cos(0) = -c_1 (0) + c_2 (1) = c_2 \end{aligned}$$

Thus, $c_1 = 1$ and $c_2 = 2$. And so the unique solution to the original 2nd order initial value problem is

$$y(x) = \cos(x) + 2\sin(x)$$

Reduction of Order

The Completeness Theorem thus reduces to the problem of solving

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{6}$$

to the problem of finding two **independent solutions** (i.e., a pair y_1, y_2 of solutions satisfying $W[y_1, y_2] \neq 0$).

In fact, the main result of Tuesday's lecture will reduced the problem of solving (6) to that of finding a **single** solution $y_1(x)$.

Theorem

Suppose $y_1(x)$ is a solution of (6). Then the function

$$y_2(x) \equiv y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\, dx\right] dx$$

will be a second independent solution of (6).