Math 2233 - Lecture 9

Agenda:

1. 2nd Order Linear ODEs, Standard Form
2. The Existence and Uniqueness Theorem
3. Differential Operator Notation
4. Homogeneous vs. Nonhomogeneous Linear ODEs
5. Two Fundamental Theorems for Homogeneous 2nd Order Linear ODEs
   - The Superposition Principle
   - The Completeness Theorem

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where

$$\begin{aligned} p(x) &= \frac{B(x)}{A(x)} \\ q(x) &= \frac{C(x)}{A(x)} \\ g(x) &= \frac{D(x)}{A(x)} \end{aligned}$$

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with v_0 and x_0 constants.

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This is, effectively, Newton's 1st Law of Motion.

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$$\begin{aligned} y\left(x_0\right) &= y_0 \\ y'\left(x_0\right) &= y_0' \end{aligned}$$

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- The E&U theorem does not address the issue of how to find solutions of a second order linear ODE.
- In fact, finding solutions to ODEs of the form (2) is going to be the main problem for the rest of the course.

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$$\begin{aligned} L[\phi] &= \left(\tfrac{d^2}{dx^2} + p(x)\tfrac{d}{dx} + q(x)\right)\phi \\ &= \tfrac{d^2\phi}{dx^2} + p(x)\tfrac{d\phi}{dx} + q(x)\phi \quad . \end{aligned}$$

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An ODE of the form (2) with $g(x) \neq 0$) is said to be **nonhomogeneous**.

The Superposition Principle

Our first results will be about solutions of homogeneous 2nd order linear ODEs.

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$$y = c_1 y_1(x) + c_2 y_2(x)$$

is also a solution to the differential equation.

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$$\begin{aligned} L[c_1y_1 + c_2y_2] \quad = \quad & \frac{d^2}{dx^2}(c_1y_1 + c_2y_2) \\ & + p(x)\frac{d}{dx}(c_1y_1 + c_2y_2) \\ & + q(x)(c_1y_1 + c_2y_2) \end{aligned}$$

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&= c_1\left(\frac{d^2 y_1}{dx^2} + p(x)\frac{dy_1}{dx} + q(x) y_1\right) \\
&\quad + c_2\left(\frac{d^2 y_2}{dx^2} + p(x)\frac{dy_2}{dx} + q(x) y_2\right)
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- As such, a lot of the results of Math 3013 Linear Algebra will find direct application in the study of linear differential equations.
- However, since Math 3013 is not a prerequisite for Math 2233, I will only make passing remarks about the connections with Linear Algebra.

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\frac{d^2 y}{dx^2} + y &= \frac{d^2}{dx^2}\left(c_1 \cos(x) + c_2 \sin(x)\right) + \left(c_1 \cos(x) + c_2 \sin(x)\right) \\
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&= -c_1 \cos(x) - c_2 \sin(x) + c_1 \cos(x) + c_2 \sin(x) \\
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Are all the solutions of (6) expressable in form (7) for some choice of c_1 and c_2?

This will not always be the case; but when **every** solution of (6) can be written as (7), we shall say that the two solutions y_1 and y_2 form a **fundamental set** of solutions if every solution of (6) can be expressed in the form (7).

The Completeness Theorem

The following theorem provides a simple way to check if a pair of solutions is a fundamental set of solutions.

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Suppose y_1 *and* y_2 *are two solutions of*

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This theorem reduces to the problem of solving (6) completely, to the task of finding two solutions, y_1 and y_2, for which (8) holds.

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So long as the denominators on the RHS of (12) and (13) are non-zero, we can thus find constants c_1 and c_2 such that an arbitrary solution $Y(x)$ and the solution $y(x) = c_1 y_1(x) + c_2 y_2(x)$ satisfy the same initial conditions And so by Uniqueness part of the E&U Theorem

$$Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$\square$

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Two solutions $y_1(x)$ and $y_2(x)$ satisfying the Wronksian Condition form a fundamental set of solution. Sometimes, we'll call two solutions satisfying the Wronkskian Condition **independent solutions**.

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2. One then uses the initial conditions to deduce two equations for the constants c_1 and c_2 appearing in the general solution.
3. Solving these equations for constants c_1 and c_2 identifies the particular solution that satisfies the initial condition, and thus solves the initial value problem.

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$$\begin{aligned} W(y_1, y_2) &\equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \\ &= \cos(x)\,(\cos(x)) - (-\sin(x))\sin(x) \\ &= 1 \\ &\neq 0 \quad . \end{aligned}$$

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Thus, $c_1 = 1$ and $c_2 = 2$. And so the unique solution to the original 2nd order initial value problem is

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The Completeness Theorem thus reduces to the problem of solving

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to the problem of finding two **independent solutions** (i.e., a pair y_1, y_2 of solutions satisfying $W[y_1, y_2] \neq 0$).

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Theorem

Suppose $y_1(x)$ is a solution of (6). Then the function

$$y_2(x) \equiv y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\, dx\right] dx$$

will be a second independent solution of (6).