Math 2233 - Lecture 10: Reduction of Order

Agenda

1. Solutions of 2nd Order, Homogeneous, Linear ODEs
   - E&U Theorem
   - Superposition Principle
   - Completeness Theorem
2. The Wronskian Condition and Linear Algebra
3. Reduction of Order
4. Examples
5. Constant Coefficients Case

2nd Order Homogeneous, Linear ODEs

Most General 2nd Order Linear ODE

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

2nd Order, Linear ODEs : Standard Form

$$y'' + p(x)y' + q(x)y = g(x)$$

Homogeneous 2nd Order, Linear ODEs : Standard Form

$$y'' + q(x)y' + p(x)y = 0$$

Fundamental Theorems

Theorem (Existence and Uniqueness Theorem)

So long as the functions $p(x)$, $q(x)$, and $g(x)$ are continuous, there exists **one and only one** *solution of*

$$y'' + p(x)\,y' + q(x)\,y = g(x) \tag{1}$$

satisfying

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y'_0 \end{aligned} \tag{2}$$

The Superposition Principle

Theorem (Superposition Principle)

If $y_1(x)$ *and* $y_2(x)$ *are two solutions of*

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{3}$$

then any function of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{4}$$

will also be a solution of (3).

The Completeness Theorem

Theorem (Completeness Theorem)

If $y_1(x)$ and $y_2(x)$ are two solutions of

$$y'' + p(x)\,y' + q(x)\,y = 0 \qquad (3)$$

such that

$$0 \neq W[y_1, y_2](x) \equiv y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x) \qquad (5)$$

Then **every solution** *of (3) can be written as*

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \qquad (6)$$

(5) is called the **Wronskian Condition**.
(6) is thus **the form of the general solution** to (3).

Solutions y_1, y_2 satisfying (5) are called a **fundamental set of solutions** (or **independent solutions**) of (3).

Geometric Interpretation of the Wronskian Condition

Digression: Independent Vectors in the Plane

Two non-zero vectors $\mathbf{A}, \mathbf{B} \in \mathbb{R}^2$ are said to be **independent** if

$$\mathbf{B} \neq \lambda \mathbf{A}$$

(i.e., **A** and **B** are neither parallel or anti-parallel).

Fact

If $\mathbf{A}, \mathbf{B} \in \mathbb{R}^2$ *are independent vectors, then* **every** *vector* $\mathbf{V} \in \mathbb{R}^2$ *can be expressed as*

$$\mathbf{V} = c_1 \mathbf{A} + c_2 \mathbf{B}$$

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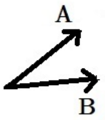

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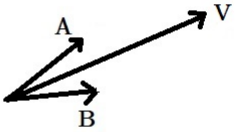

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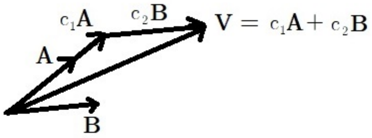

Connection with the Wronskian Condition

Now recall the Wronskian Condition

$$0 \neq W[y_1, y_2](x) = y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)$$

Suppose this condition **does not hold** for two functions $y_1(x)$ and $y_2(x)$; i.e.,

$$y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x) = 0 \tag{5}$$

Then we can think of (5) as a differential equation for $y_2(x)$ (with $y_1(x)$ a given function). After dividing (5) by $y_1(x)$

$$y_2' - \frac{y_1'(x)}{y_1(x)} y_2 = 0$$

This is a first order linear ODE in the standard form

$$y' + P(x)\, y' = G(x)$$

with $P(x) = \frac{y_1'(x)}{y_1(x)}$ and $G(x) = 0$. So we can readily solve it.

Connection with the Wronskian Condition, Cont'd

Noting that

$$\frac{d}{dx}\left(\ln\left|y_1(x)\right|\right) = \frac{1}{y_1(x)} y_1'(x) = -P(x)$$

$$\begin{aligned}
\mu(x) &= \exp\left[\int P(x)\,dx\right] \\
&= \exp\left[-\int \frac{d}{dx}\left(\ln\left|y_1(x)\right|\right)dx\right] \\
&= \exp\left[-\ln\left|y_1(x)\right|\right] \\
&= \frac{1}{y_1(x)}
\end{aligned}$$

and so

$$\begin{aligned}
y_2(x) &= \frac{1}{\mu(x)} \int \mu(x)\, G(x)\,dx + \frac{C}{\mu(x)} \\
&= y_1(x) \int \frac{1}{y_1(x)}(0)\,dx + C y_1(x) \\
&= 0 + C y_1(x)
\end{aligned}$$

Connection with the Wronskian Condition, Cont'd

We have just proved

Lemma

If $y_1(x)$ and $y_2(x)$ fail to satisfy the Wronskian condition, then

$$y_2(x) = \lambda y_1(x)$$

for some constant λ.

The analogy:

- Let $\mathbf{A}, \mathbf{B}$ be vectors in the plane, so long as

$$\mathbf{B} \neq \lambda \mathbf{A}$$

every vector $\mathbf{V}$ in the plane $\mathbb{R}^2$ can be expressed as

$$\mathbf{V} = c_1 \mathbf{A} + c_2 \mathbf{B}$$

- Let $y_1(x)$, $y_2(x)$ be two solutions of

$$y'' + p(x)\, y' + q(x)\, y = 0 \qquad (*)$$

If $y_1(x)$ and $y_2(x)$ satisfy the Wronskian Condition, then

$$y_2(x) \neq \lambda y_1(x)$$

and **every** solution of (*) can be expressed as

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Reduction of Order

Recall that the general solution of a second order homogeneous linear differential equation

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

is given by

$$y(x) = c_1y_1(x) + c_2y_2(x) \tag{2}$$

where y_1 and y_2 are any two solutions such that

$$0 \neq W[y_1, y_2](x) = y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{3}$$

I'll now show you how to find the general solution of (1) starting with just a single solution of (1).

Hypothesis:
Suppose we have one non-trivial solution $y_1(x)$ of (1) and suppose there is another solution of the form

$$y_2(x) = v(x)y_1(x) \quad .$$

(Here we are making an "ansatz" for a second solution.) Then

$$\begin{aligned} W[y_1, y_2] &= y_1 y_2' - y_1' y_2 \\ &= y_1(v' y_1 + v y_1') - y_1'(v y_1) \\ &= (y_1)^2 v' \\ &\neq 0 \end{aligned}$$

unless $v' = 0$.
Thus, any solution we construct by multiplying our given solution $y_1(x)$ by a non-constant function $v(x)$ will give us another linearly independent solution.

So let's look for a second solution indirectly, by finding a function $v(x)$ so that $v(x)y_1(x)$ is a solution.
Inserting $y(x) = v(x)y_1(x)$ into (1):

$$\begin{aligned} 0 &= \tfrac{d^2}{dx^2}(vy_1) + p(x)\tfrac{d}{dx}(vy_1) + q\,(vy_1) \\ &= v''y_1 + 2v'y_1' + vy_1'' + p(x)v'y_1 + p(x)vy_1' + qvy_1 \\ &= v\,(y_1'' + p(x)y_1' + q(x)y_1) + v''y_1 + (2y_1' + p(x)y_1)\,v' \end{aligned}$$

The first term vanishes since y_1 is a solution of (1), so $v(x)$ must satisfy

$$0 = y_1 v'' + \left(2y_1' + p(x)y_1\right) v' \tag{4}$$

or

$$v'' + \left(p(x) + \frac{2y_1'}{y_1}\right) v' = 0 \quad . \tag{5}$$

Note that this is a first order linear ODE for $v'(x)$.

So set

$$u(x) = v'(x) \quad . \tag{6}$$

Then (5) becomes

$$u' + \left(p(x) + \frac{2y_1'(x)}{y_1(x)} \right) u = 0 \quad . \tag{7}$$

The integrating factor for this 1st Order Linear ODE is

$$\begin{aligned}
\mu(x) &= \exp\left[\int \left(p(x) + \frac{2y_1'(x)}{y_1(x)} \right) dx \right] \\
&= \exp\left[\int \frac{2y_1'(x)}{y_1(x)} \right] \exp\left[\int p(x)\, dx \right] \\
&= \exp\left[2 \int \left(\frac{d}{dx} (\ln |y_1(x)|) \right) dx \right] \exp\left[\int p(x)\, dx \right] \\
&= \exp\left[2 \ln \left[y_1(x) \right] \right] \exp\left[\int p(x)\, dx \right] \\
&= (y_1(x))^2 \exp\left[\int p(x)\, dx \right]
\end{aligned}$$

And so

$$
\begin{aligned}
u(x) &= \frac{1}{\mu(x)} \int \mu(x)(0)\, dx + \frac{C}{\mu(x)} \\
&= 0 + C \frac{1}{(y_1(x))^2} \exp\left[- \int p(x)\, dx \right]
\end{aligned}
$$

and so

$$
u(x) = \frac{C}{(y_1(x))^2} \exp\left[- \int^x p(t) dt \right] \quad .
$$

Now recall from (6) that $u(x)$ is the derivative of the factor v(x) which we originally sought out to find. (recall $y_2(x) = v(x)y_1(x)$ will be our second solution)
So

$$\begin{aligned} v(x) &= \int^x u(t)\, dt + D \\ &= \int^x \left[\frac{C}{(y_1(t))^2} \exp\left[-\int^t p(t')dt' \right] \right] + D \end{aligned}$$

Since we only need one 2 second solution, we can take $C = 1$ and $D = 0$.

So given one solution $y_1(x)$ of (1), a second solution $y_2(x)$ of (1) can be formed by computing

$$v(x) = \int^x u(t)\, dt$$

where

$$u(x) = \frac{1}{(y_1(x))^2} \exp\left[-\int^x p(t) dt\right]$$

and then setting

$$y_2(x) = v(x) y_1(x) \quad .$$

The general solution of (1) is then

$$y(x) = c_1 y_1(x) + c_2 v(x) y_1(x) \quad .$$

This technique for constructing the general solution from single solution of a second order linear homogeneneous differential equation is called **Reduction of Order**.
Summarizing

Theorem (Reduction of Order)

If $y_1(x)$ *is a solution of*

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

then a 2nd independent solution $y_2(x)$ *can be calculated via the formula*

$$\boxed{y_2(x) = y_1(x)\int^x \frac{1}{(y_1(s))^2}\exp\left[-\int^s p(t)dt\right] ds}$$

Once y_2 has been calculated, the general solution to (1) can be written

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Example 1

$$y_1(x) = e^{-x}$$

is one solution of

$$y'' + 2y' + y = 0 \quad .$$

Find another linearly independent solution and then write down the general solution.
Well, $p(x) = 2$, so

$$\begin{aligned} u(x) &= \tfrac{1}{(y_1(x))^2} \exp\left[-\int^x p(t)dt\right] \\ &= \tfrac{1}{e^{-2x}} \exp\left[-2x\right] \\ &= e^{2x}e^{-2x} \\ &= 1 \end{aligned}$$

So

$$v(x) = \int^x u(t)\,dt = \int^x dt = x.$$

Example 1, Cont'd

Thus,

$$y_2(x) = v(x)y_1(x) = xe^{-x} \quad .$$

and so the general solution to the original ODE will be

$$y(x) = c_1 e^{-x} + c_2 x e^{-x}$$

Example 2

$$y_1(x) = x$$

is a solution of

$$x^2y'' + 2xy' - 2y = 0 \quad .$$

This time we'll use the Reduction of Order formula

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\,dx\right] dx$$

(where I'm being a bit sloppy with respect to the variables of integration).
Putting the DE in standard form

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0$$

we see

$$p(x) = \frac{2}{x} \qquad , \qquad q(x) = -\frac{2}{x^2}$$

We can now calculate a second solution

$$
\begin{aligned}
y_2(x) &= y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\, dx\right] dx \\
&= x \int \frac{1}{x^2} \exp\left[-\int \frac{2}{x} dx\right] dx \\
&= x \int \frac{1}{x^2} \exp\left[-2 \ln |x|\right] dx \\
&= x \int \frac{1}{x^2} x^{-2} dx \\
&= x \int x^{-4} dx \\
&= x \left(-\frac{1}{3} x^{-3}\right) \\
&= -\frac{1}{3} x^{-2}
\end{aligned}
$$

The general solution to the original differential equation is thus

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_2 y_2(x) \\ &= c_1 x + c_2 \left(-\frac{1}{3} x^{-2} \right) \\ &= c_1 e^{-x} + c_2 x^{-2} \end{aligned}$$

In the last line, the constant factor of $-\frac{1}{3}$ has been "absorbed" into the arbitrary constant c_2 (for the sake of tidyness).

Second Order Linear Equations with Constant Coefficients

We are now ready to actually solve some linear ODE's of degree 2 **from scratch**.

We shall begin with differential equations of a particularly simple type; equations of the form

$$ay'' + by' + cy = 0 \tag{9}$$

where a, b and c are constants.

A clue as to how one might construct a solution to (9) comes from the observation that (9) implies that y'', y' and y are related to one another by multiplicative constants.

But this is a property of exponential functions; i.e., a functions of the form

$$y(x) = e^{\lambda x} \tag{10}$$

have the property that

$$y' = \lambda e^{\lambda x} = \lambda y \qquad \text{and} \qquad y'' = \lambda^2 e^{\lambda x} = \lambda^2 y$$

We will therefore look for solutions of (9) having the form (10).

Constant Coefficients Case, Cont'd

Plugging (10) into (9) yields

$$0 = a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} = \left(a\lambda^2 + b\lambda + c\right) e^{\lambda x} \quad .$$

Since the exponential function $e^{\lambda x}$ never vanishes (for finite x) we must have

$$a\lambda^2 + b\lambda + c = 0 \quad .$$

Equation (11) is called the **characteristic equation** for (9). Any number λ satisfying the characteristic equation will give us a solution $y(x) = e^{\lambda x}$ of the orginal differential equation (9). Now because (11) is a quadratic equation we can employ the Quadratic Formula to find all of its roots:

$$a\lambda^2 + b\lambda + c = 0 \qquad \Rightarrow \qquad \lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad .$$

However, a root λ of (11) need not be a real number.
We shall postpone until next week the case when the roots of (11) are complex numbers.
For simplicity, and just for today, we'll assume that $\sqrt{b^2 - 4ac}$ is a positive real number and so

$$\begin{aligned} \lambda_+ &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ \lambda_- &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned}$$

are distinct real roots of (11). Thus,

$$\begin{aligned} y_1(x) &= e^{\lambda_+ x} \\ y_2(x) &= e^{\lambda_- x} \end{aligned}$$

will both be solutions of (9).
Note that

$$\begin{aligned} W[y_1, y_2](x) &= \left(e^{\lambda_+ x}\right)\left(e^{\lambda_- x}\right)' - \left(e^{\lambda_+ x}\right)'\left(e^{\lambda_- x}\right) \\ &= (\lambda_- - \lambda_+)\, e^{(\lambda_+ + \lambda_-)x} \\ &\neq 0 \end{aligned}$$

Constant Coefficients Case, Cont'd

And so the solutions $y_1 = e^{\lambda_+ x}$ and $y_2 = e^{\lambda_- x}$ will form a fundamental set of solutions.
The general solution to

$$ay'' + by' + cy = 0 \tag{9}$$

will thus be

$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}$$

Method for Constant Coefficients Case

$$ay'' + by' + cy = 0 \tag{12}$$

1. Substitute "trial solution" $y(x) = e^{\lambda x}$ into (9).
2. Divide result by $e^{\lambda x}$ to get the **characteristic equation** for (9)
$$a\lambda^2 + b\lambda + c = 0 \tag{13}$$
3. Solve (13) either by factoring the LHS or via the Quadratic Formula
$$\lambda_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad , \quad \lambda_- = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$
4. So long as $b^2 - 4ac > 0$, the roots λ_+ and λ_- will be distinct real numbers, and the functions $y_1 = e^{\lambda_+ x}$ and $y_2 = e^{\lambda_- x}$ will form a fundamental set of solutions. The general solution will then be
$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}$$

What if $b^2 - 4ac \leq 0$?

- If $b^2 - 4ac = 0$, then
  $\lambda_+ = \lambda_- \quad \Rightarrow \quad y_1(x) = y_2(x) \quad \Rightarrow \quad$ only 1 independent solution
  We need two independent solutions for to write down the general solution.
- If $b^2 - 4ac < 0$, then $\lambda_\pm$ are complex numbers. What is $e^{\lambda x}$ when $\lambda \in \mathbb{C}$?

We'll resolve these two issues in the next lecture.

Example 3

$$y'' + 3y' + 2y = 0 \quad .$$

Setting $y(x) = e^{\lambda x}$ and plugging into the differential equation we get

$$\begin{aligned} 0 &= \lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} + 2e^{\lambda x} \\ &= e^{\lambda x}\left(\lambda^2 + 3\lambda + 2\right) \\ &= e^{\lambda x}(\lambda + 1)(\lambda + 2) \end{aligned}$$

Since $e^{\lambda x}$ never vanishes for any finite x, we must have

$$\lambda = -1 \quad \text{or} \quad \lambda = -2 \quad .$$

We thus find two distinct solutions

$$\begin{aligned} y_1(x) &= e^{-x} \\ y_2(x) &= e^{-2x} \quad . \end{aligned}$$

Example 3, Cont'd

Note that

$$\begin{aligned} W\left[y_1, y_2\right] &= \left(e^{-x}\right)\left(\frac{d}{dx}e^{-2x}\right) - \left(\frac{d}{dx}e^{-x}\right)\left(e^{-2x}\right) \\ &= \left(e^{-x}\right)\left(-2e^{-2x}\right) - \left(-e^{-x}\right)\left(e^{-2x}\right) \\ &= \left(-2+1\right)e^{-3x} \\ &\neq 0 \end{aligned}$$

and so $y_1(x) = e^{-x}$ and $y_2(x) = e^{-2x}$ are independent solutions. The general solution is thus

$$y(x) = c_1 e^{-x} + c_2 e^{-2x} \quad .$$