Math 2233 - Lecture 10: Reduction of Order

Agenda

1. Solutions of 2nd Order, Homogeneous, Linear ODEs
   - E&U Theorem
   - Superposition Principle
   - Completeness Theorem
2. The Wronskian Condition and Linear Algebra
3. Reduction of Order
4. Examples
5. Constant Coefficients Case

2nd Order Homogeneous, Linear ODEs

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Most General 2nd Order Linear ODE

$$a(x)y'' + b(x)y' + c(x)y = d(x)$$

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$$y'' + p(x)y' + q(x)y = g(x)$$

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Homogeneous 2nd Order, Linear ODEs : Standard Form

$$y'' + q(x)y' + p(x)y = 0$$

Fundamental Theorems

Theorem (Existence and Uniqueness Theorem)

So long as the functions $p(x)$, $q(x)$, *and* $g(x)$ *are continuous, there exists* **one and only one** *solution of*

$$y'' + p(x)\, y' + q(x)\, y = g(x) \tag{1}$$

satisfying

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y'_0 \end{aligned} \tag{2}$$

The Superposition Principle

Theorem (Superposition Principle)

If $y_1(x)$ *and* $y_2(x)$ *are two solutions of*

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{3}$$

then any function of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{4}$$

will also be a solution of (3).

The Completeness Theorem

Theorem (Completeness Theorem)

If $y_1(x)$ *and* $y_2(x)$ *are two solutions of*

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such that

$$0 \neq W[y_1, y_2](x) \equiv y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x) \tag{5}$$

Then **every solution** *of (3) can be written as*

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Solutions y_1, y_2 satisfying (5) are called a **fundamental set of solutions** (or **independent solutions**) of (3).

Geometric Interpretation of the Wronskian Condition

Digression: Independent Vectors in the Plane

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Two non-zero vectors $\mathbf{A}, \mathbf{B} \in \mathbb{R}^2$ are said to be **independent** if

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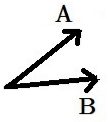

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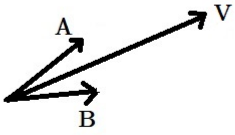

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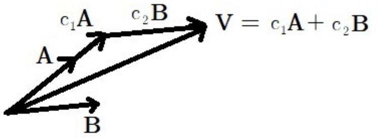

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$$\frac{d}{dx}\left(\ln\left|y_1\left(x\right)\right|\right) = \frac{1}{y_1\left(x\right)}y_1'\left(x\right) = -P\left(x\right)$$

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$$\begin{aligned}y_2(x) &= \frac{1}{\mu(x)}\int \mu(x)\,G(x)\,dx+\frac{C}{\mu(x)}\\ &= y_1(x)\int \frac{1}{y_1(x)}(0)\,dx+Cy_1(x)\\ &= 0+Cy_1(x)\end{aligned}$$

Connection with the Wronskian Condition, Cont'd

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Lemma

If $y_1(x)$ and $y_2(x)$ fail to satisfy the Wronskian condition, then

$$y_2(x) = \lambda y_1(x)$$

for some constant λ.

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I'll now show you how to find the general solution of (1) starting with just a single solution of (1).

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Thus, any solution we construct by multiplying our given solution $y_1(x)$ by a non-constant function $v(x)$ will give us another linearly independent solution.

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$$0 = \frac{d^2}{dx^2}(vy_1) + p(x)\frac{d}{dx}(vy_1) + q(vy_1)$$

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Note that this is a first order linear ODE for $v'(x)$.

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v(x) &= \int^x u(t)\,dt + D \\
&= \int^x \left[\frac{C}{(y_1(t))^2} \exp\left[-\int^t p(t')dt'\right]\right] + D
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Theorem (Reduction of Order)

If $y_1(x)$ is a solution of

$$y'' + p(x)y' + q(x)y = 0 \tag{1}$$

then a 2nd independent solution $y_2(x)$ can be calculated via the formula

$$\boxed{y_2(x) = y_1(x)\int^x \frac{1}{(y_1(s))^2}\exp\left[-\int^s p(t)dt\right]ds}$$

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Once y_2 has been calculated, the general solution to (1) can be written

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

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Example 1, Cont'd

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$$p(x) = \frac{2}{x} \qquad , \qquad q(x) = -\frac{2}{x^2}$$

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In the last line, the constant factor of $-\frac{1}{3}$ has been "absorbed" into the arbitrary constant c_2 (for the sake of tidyness).

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We'll resolve these two issues in the next lecture.

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Note that

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