Math 2233 - Lecture 11 : Homogeneous Linear ODEs with Constant Coefficients

Agenda:

1. Homogeneous Linear ODEs with Constant Coefficients
2. Difficulties with Roots of the Characteristic Equation
   - What to do when the Characteristic Equation has only one solution
   - What to do when the Characteristic Equation has only complex numbers as solutions
3. Summary of the 3 basic cases
4. Examples

Solutions of Homogeneous ODEs

Recall that the **general solution** of a 2^{nd} order linear homogeneous differential equation

$$L[y] = y'' + p(x)y' + q(x)y = 0 \tag{1}$$

is always a linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{2}$$

of two linearly independent solutions y_1 and y_2, Here "independent" means

$$0 \neq W[y_1, y_2](x) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{3}$$

Homogeneous Equations with Constant Coefficients

Let us now return to the special case of a homogeneous second order linear differential equation with constant coefficients; i.e., differential equations of the form

$$ay'' + by' + cy = 0 \tag{4}$$

where a, b and c are constants.
We saw in the last lecture that one can construct solutions of the differential equation (4) by looking for solutions of the form

$$y(x) = e^{\lambda x} \quad . \tag{*}$$

Let us recall that construction. Plugging (*) into (4) yields

$$0 = a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + ce^{\lambda x} = \left(a\lambda^2 + b\lambda + c\right) e^{\lambda x} \quad .$$

Since the exponential function $e^{\lambda x}$ never vanishes we must have

$$a\lambda^2 + b\lambda + c = 0 \quad . \tag{5}$$

The Characteristic Equation

The equation

$$a\lambda^2 + b\lambda + c = 0 \quad . \tag{5}$$

is called the **characteristic equation** ("auxiliary equation" in the text) for the differential equation

$$ay'' + by' + cy = 0 \tag{4}$$

Each solution λ of the characteristic equation can be used to construct an exponential function $y(x) = e^{\lambda x}$ that will be a solution of (4). Now because (5) is a quadratic equation, we can employ the Quadratic Formula to find all of its roots:

$$a\lambda^2 + b\lambda + c = 0 \qquad \Rightarrow \qquad \lambda = \lambda_{\pm} \equiv \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Using the two roots λ_+ and λ_- of the characteristic equation, we can now write down two associated solutions of (4)

$$y_1(x) = e^{\lambda_+ x} \qquad , \qquad y_2(x) = e^{\lambda_- x}$$

But are $y_1(x)$ and $y_2(x)$ are independent solutions of (4)?
Well, if

$$\begin{aligned} 0 &\neq W(y_1, y_2) \\ &= y_1 y_2' - y_1' y_2 \\ &= \lambda_- e^{\lambda_+ x} e^{\lambda_- x} - \lambda_+ e^{\lambda_+ x} e^{\lambda_- x} \\ &= (\lambda_- - \lambda_+)\, e^{(\lambda_+ + \lambda_-)x} \\ &= \frac{\sqrt{b^2 - 4ac}}{a} e^{-\frac{b}{a}x} \end{aligned}$$

And so we can conclude that if $b^2 - 4ac \neq 0$, then the roots (6) furnish two linearly independent solutions of (4)

The general solution will thus be

$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}$$

Summarizing:

$$
\begin{array}{rcl}
ay'' + by' + c & = & 0 \qquad \text{(constant coeff. ODE)} \\
& \Downarrow & \qquad \text{(try looking for solutions of the form } e^{\lambda x}\text{)} \\
a\lambda^2 + b\lambda + c & = & 0 \qquad \text{(Ch. Eq)} \\
& \Downarrow & \qquad \text{(Quadratic formula)} \\
\lambda & = & \lambda_\pm = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(roots of the Ch. Eq.)} \\
& \Downarrow & \\
y_1(x) & = & e^{\lambda_+ x} \quad , \quad y_2 = e^{\lambda_- x} \qquad \text{(indep. solutions if } \lambda_+ \neq \lambda_-\text{)} \\
& \Downarrow & \\
y(x) & = & c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x} \qquad \text{(general solution of the ODE)}
\end{array}
$$

Caveats to this Procedure

Looking more closely at our formula for the roots of the characteristic equation

$$\lambda_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (*)$$

we see that this procedure runs into some obstacles if $b^2 - 4ac \leq 0$

- If $b^2 - 4ac = 0$, then $\lambda_+ = \lambda_-$ and so our two solutions $y_1(x) = e^{\lambda_+ x}$ and $y_2(x) = e^{\lambda_- x}$ coincide. So in this case, we only get 1 independent solution.
  But we need 2 independent solutions in order to write down the general solution of (4). What do we do to complete the solution?
- If $b^2 - 4ac < 0$, then to apply the formula (*) above, we have to take the square root of a negative number.
  This is going to mean that the numbers $\lambda_{\pm}$ kicked out by the Quadratic Formula are going to be complex numbers.
  What is $e^{\lambda x}$ if λ is a complex number?

Let's consider, in detail now, the various possibilities, case-by-case.

Case (i): $b^2 - 4ac > 0$

Because $b^2 - 4ac$ is positive, $\sqrt{b^2 - 4ac}$ is a positive real number and

$$\begin{aligned} \lambda_+ &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ \lambda_- &= \frac{-b-\sqrt{b^2-4ac}}{2a} \end{aligned} \tag{6}$$

are distinct real roots of (5). Thus,

$$\begin{aligned} y_1 &= e^{\lambda_+ x} \\ y_2 &= e^{\lambda_- x} \end{aligned}$$

will be independent solutions of (4).
The general solution will be

$$y(x) = c_1 e^{\lambda_+ x} + c_2 e^{\lambda_- x}$$

(No caveats in this case.)

Case (ii): $b^2 - 4ac = 0$

If $b^2 - 4ac = 0$, however, the characteristic equation only gives us one distinct root; because in this case

$$\lambda_+ = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \lambda_-$$

and so

$$y_1(x) = e^{-\frac{b}{2a}x} = y_2(x) \quad \Rightarrow \quad W[y_1, y_2](x) = 0$$

So we have not yet found two independent solutions.

To find a second fundamental solution, we can use the method of Reduction of Order.

Using Reduction of Order to find a 2nd solution

Theorem (Reduction of Order)

If $y_1(x)$ is a solution of $y'' + p(x)y' + q(x)y = 0$, a second independent solution can be calculated as

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[\int^s -p(t)dt\right] ds$$

In the case at hand, the ODE in standard form is

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$$

and $y_1(x) = e^{-\frac{b}{2a}x}$ is the solution corresponding to the root $\lambda = -\frac{b}{2a}$. In the next slide, we'll plug in $y_1(x) = e^{-\frac{b}{2a}x}$ and $p(x) = \frac{-b}{a}$ into the Reduction of Order formula to calculate a second independent solution.

Reduction of Order Calculation

$$\begin{aligned}
y_2(x) &= e^{-\frac{b}{2a}x}\int^x \frac{1}{\left(e^{-\frac{b}{2a}s}\right)^2}\exp\left[\int^s -\frac{b}{a}dt\right]ds \\
&= e^{-\frac{b}{2a}x}\int^x \frac{1}{e^{-\frac{b}{a}s}}\exp\left[-\frac{b}{a}s\right]ds \\
&= e^{-\frac{b}{2a}x}\int^x e^{\frac{b}{a}s}e^{-\frac{b}{a}s}ds \\
&= e^{-\frac{b}{2a}x}\int^x ds \\
&= xe^{-\frac{b}{2a}x} \\
&= xy_1(x)
\end{aligned}$$

Summary of Case (ii) where $b^2 - 4ac = 0$

When $b^2 - 4a = 0$

- we only have one root of the characterisitic equation
- so we obtain only 1 independent exponential solution $y_1(x) = e^{-\frac{b}{2a}x}$ of the original ODE.
- To get a second independent solution, $y_2(x)$, we must use Reduction of Order.
- However, in the case when $b^2 - 4ac = 0$, the Reduction of Order calculation always produces a second solution of the form
$$y_2(x) = xy_1(x) = xe^{-\frac{b}{2a}x}$$

Thus, when $b^2 - 4ac = 0$, the general solution will be

$$y(x) = c_1 e^{-\frac{b}{2a}x} + c_2 xe^{-\frac{b}{2a}x}$$

Case (iii): $b^2 - 4ac < 0$

In this case

$$\sqrt{b^2 - 4ac}$$

will be undefined unless we introduce complex numbers. But once we introduce the pure imaginary number i, defined by $\sqrt{-1} = i$, we have

$$\sqrt{b^2 - 4ac} = \sqrt{(-1)(4ac - b^2)} = \sqrt{-1}\sqrt{4ac - b^2} = i\sqrt{4ac - b^2} \quad .$$

Note how the square root on the far right hand side is well-defined since $4q - p^2$ is a positive number. Thus,

$$\lambda_{\pm} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \alpha \pm i\beta$$

where

$$\alpha = -\frac{b}{2a} \quad , \quad \beta = \frac{\sqrt{4ac - b^2}}{2a} \quad ,$$

will be complex solutions of the characteristic equation (4).

The Main Question for Case (iii)

So how do we use the complex roots $\lambda_{\pm} = \alpha \pm i\beta$ of the characteristic equation to get functions that are solutions of the differential equations?

We'll answer this by working out how to make sense of functions of the form

$$y(x) = e^{(\alpha + i\beta)x}$$

So what is $e^{ax+i\beta x}$?

We will now address the problem of ascribing some calculable meaning to

$$e^{\alpha x+i\beta x}$$

as a function of x.
Let's begin by recalling a little bit about the complex numbers. A complex number z is an expression of the form

$$z = x + iy \qquad (*)$$

where x and y are ordinary (real) numbers, and $i = \sqrt{-1}$ has the property that $i^2 = -1$

The real number x on the R.H.S. of (*) is called the **real part** of z (often written $x = Re\,(z)$) and the real number y is called the **imaginary part** of z (denoted $Im\,(z)$). Thus, we specify complex numbers by prescribing their real and imaginary parts.

Polynomial Functions of a Complex Variable

The simple rule $i^2 = -1$ is all we need in order to carry out complex arithemetic. For example, to multiply two complex numbers $z = a + ib$, and $z' = c + id$

$$zz' = (a + ib)(c + id) = ac + ibd + iad + i^2 bd = (ac - bd) + i(ad + bc)$$

In this way, we view complex arithmetic as an extension of the ordinary arithmetic of the real numbers.
Having a notion of complex arithmetic, we can now convert a polynomial $p(x)$ in a real variable x into a complex-valued function. If

$$p(x) = a_n x^n + \cdots + a_1 x + a_0$$

then

$$p(z) = \text{the complex number found by evaluating } a_n z^n + \cdots + a_1 z + a_0 \text{ using complex arithmetic}$$

Example: Converting a polynomial to a complex function

Let

$$\begin{aligned} p(x) &= x^2 - 1 \\ \Rightarrow \quad p(z) &= z^2 - 1 \\ \Rightarrow \quad p(x+iy) &= (x+iy)^2 - 1 \\ &= \left(x^2 - y^2 - 1\right) + i2xy \end{aligned}$$

Complex Exponential Functions

Okay, so any polynomial function $p(x)$ can be converted into a function of a complex variable z taking values in $\mathbb{C}$.

What about exponential functions of the form $e^{(\alpha+i\beta)x}$?

Well, what we'll do is try to think of such functions as kinda like polynomials.

We'll use the idea of Taylor expansions to make such an interpretation plausible.

The Taylor Expansion of e^x

Recall from Calculus II, the idea of Taylor expansions. Any nice differentiable function $f(x)$ has a Taylor series about $x = 0$

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

Note how we can interpret the right hand side of the Taylor expansion as a sort of infinite polynomial. This will be the key idea here.

Let's look at the Taylor expansion of the exponential function. This is easy to compute since

$$\frac{d}{dx}e^x = e^x \quad \Rightarrow \quad \frac{d^n}{dx^n}e^x = e^x \quad \Rightarrow \quad \left.\frac{d^n}{dx^n}e^x\right|_{x=0} = 1 \quad \text{for all } n$$

The Taylor Expansion of e^x, Cont'd

Thus,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

This suggests setting

$$e^z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots$$

And this works very well.
In fact, using this definition of e^z, one arrives at a function of a complex variable z that retains all of the nice properties of the usual exponential function.
In particular

$$e^{z+z'} = e^z e^{z'}$$

Now let's look at e^z as e^{x+iy}. We have

$$e^{x+iy} = e^x e^{iy}$$

Now the factor e^x is just the exponential function of the real variable x. The mysterious part is the purely imaginary exponential factor e^{iy}. So let's look more closely at that

$$e^{iy} = 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \qquad (*)$$

Now because

$$\begin{aligned} i^0 &= 1 \\ i^1 &= i \\ i^2 &= -1 \\ i^3 &= i^2(i) = -i \\ i^4 &= \left(i^2\right)\left(i^2\right) = (-1)(-1) = 1 \end{aligned}$$

we see that powers of i end up being reducible to either ± 1 or $\pm i$.

In fact, the even powers of i will always be ± 1 and the odd powers of i will be $\pm i$. This then allows us to split the right hand side of (*) into even powered terms that are purely real, and the odd powers that will be purely imaginary.
One finds

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots\right)$$

And here a miracle occurs. It turns out the Taylor expansion of the cosine function is

$$\cos(y) = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right)$$

and the Taylor expansion of the sine function is

$$\sin(y) = y - \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots$$

Making these identifications, we then have

Theorem (The Euler Formula)

$$\boxed{e^{iy} = \cos(y) + i\sin(y)}$$

Now you can compute, for example, e^{3i} on your calculator. It will be

$$e^{3i} = \cos(3) + i\sin(3)$$

We can also compute $e^z = e^{x+iy}$ as

$$e^{x+iy} = e^x e^{iy} = e^x\cos(y) + ie^x\sin(y) \quad .$$

And we can compute $e^{(\alpha+i\beta)x}$ as

$$e^{(\alpha+i\beta)x} = e^{\alpha x}e^{i\beta x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x)$$

Returning to Case (iii): $ay'' + by' + cy = 0$ with $b^2 - 4ac < 0$

The solutions of the characteristic equation are complex numbers

$$\lambda_{\pm} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

We'll write this as

$$\lambda_{\pm} = \alpha \pm i\beta$$

where

$$\alpha = \frac{-b}{2a} \qquad , \qquad \beta = \frac{\sqrt{4ac - b^2}}{2a}$$

The corresponding solutions to the ODE will be of the form

$$\begin{aligned} \widetilde{y}_1(x) &= e^{(\alpha + i\beta)x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x) \\ \widetilde{y}_2(x) &= e^{(\alpha - i\beta)x} = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(-\beta x) \end{aligned}$$

Constructing Real-Valued Solutions

Using the Superposition Principle, we can take linear combinations of these two solutions to produce more solutions; in fact, solutions that allow us to completely avoid using complex numbers.
Set

$$\begin{aligned} y_1(x) &\equiv \frac{1}{2}\widetilde{y}_1(x) + \frac{1}{2}\widetilde{y}_2(x) \\ &= \frac{1}{2}e^{\alpha x}(\cos(x) + \cos(x)) + i\frac{1}{2}e^{\alpha x}(\sin(\beta x) + \sin(-\beta x)) \\ &= e^{\alpha x}\cos(x) + \frac{i}{2}e^{\alpha x}(\sin(\beta x) - \sin(\beta x)) \\ &= e^{ax}\cos(x) \end{aligned}$$

where in the third step we used the fact that $\sin(x)$ is an odd function of x

$$\sin(-x) = -\sin(x) \qquad \text{for all } x$$

Similarly, we set

$$\begin{aligned} y_2(x) &\equiv \frac{1}{2i}\widetilde{y}_1(x) - \frac{1}{2i}\widetilde{y}_2(x) \\ &= \frac{e^{\alpha x}}{2i}\left(\cos(\beta x) + i\sin(\beta x) - \cos(\beta x) - i\sin(-\beta x)\right) \\ &= \frac{e^{\alpha x}}{2i}\left(0 + 2i\sin(\beta x)\right) \\ &= e^{\alpha x}\sin(\beta x) \end{aligned}$$

Thus, we have obtained from our two complex valued solutions, two independent real valued solutions.
We will adopt these two real-valued solutions as our fundamental solutions to the original differential equation.

Summary: Case (iii) where $b^2 - 4ac < 0$

We have thus seen that when $b^2 - 4ac < 0$, the solutions to the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

are complex numbers

$$\lambda_\pm = \alpha \pm i\beta$$

we'll have the following two real-valued solutions to $ay'' + by' + cy = 0$

$$\begin{aligned} y_1(x) &= e^{\alpha x}\cos(\beta x) \\ y_2(x) &= e^{\alpha x}\sin(\beta x) \end{aligned}$$

and so the general solution to the differential equation will be

$$y(x) = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$$

Summary: Solving $ay'' + by' + cy = 0$

(i) When $b^2 - 4ac > 0$

- Characteristic equation has 2 distinct real roots λ_1 and λ_2
- General solution : $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

(ii) When $b^2 - 4ac = 0$

- Characteristic equation has only distinct real root $\lambda = -\frac{b}{2a}$
- General solution : $y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$

(ii) When $b^2 - 4ac < 0$

- Characteristic equation has 2 distinct complex roots $\alpha + i\beta$ and $\alpha - i\beta$
- General solution : $y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$

Example 1

The differential equation

$$y'' - 2y' - 3y$$

has as its characteristic equation

$$\lambda^2 - 2\lambda - 3 = 0 \quad .$$

The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{2 \pm \sqrt{4+12}}{2} \\ &= 3, -1 \quad . \end{aligned}$$

These are distinct real roots, so the general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-x} \quad .$$

Example 2

The differential equation

$$y'' + 4y' + 4y = 0$$

has

$$\lambda^2 + 4\lambda + 4 = 0$$

as its characteristic equation. The roots of the characteristic equation are given by

$$\begin{aligned} \lambda &= \tfrac{-4 \pm \sqrt{16-16}}{2} \\ &= -2 \quad . \end{aligned}$$

Thus, we have only one distinct root λ and the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} \quad .$$

Example 3

The differential equation

$$y'' + y' + y = 0$$

has

$$\lambda^2 + \lambda + 1 = 0$$

as its characteristic equation. The roots of the characteristic equation are

$$\begin{aligned} \lambda &= \tfrac{-1 \pm \sqrt{1-4}}{2} \\ &= -\tfrac{1}{2} \pm i \tfrac{\sqrt{3}}{2} \end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) \quad .$$