Math 2233 - Lecture 11 : Homogeneous Linear ODEs with Constant Coefficients

Agenda:

1. Homogeneous Linear ODEs with Constant Coefficients
2. Difficulties with Roots of the Characteristic Equation
   - What to do when the Characteristic Equation has only one solution
   - What to do when the Characteristic Equation has only complex numbers as solutions
3. Summary of the 3 basic cases
4. Examples

Solutions of Homogeneous ODEs

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$$0 \neq W[y_1, y_2](x) \equiv y_1(x)y_2'(x) - y_1'(x)y_2(x) \tag{3}$$

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Let's consider, in detail now, the various possibilities, case-by-case.

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(No caveats in this case.)

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To find a second fundamental solution, we can use the method of Reduction of Order.

Using Reduction of Order to find a 2nd solution

Theorem (Reduction of Order)

If $y_1(x)$ is a solution of $y'' + p(x)\, y' + q(x)\, y = 0$, a second independent solution can be calculated as

$$y_2(x) = y_1(x) \int^x \frac{1}{(y_1(s))^2} \exp\left[\int^s -p(t)dt\right] ds$$

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and $y_1(x) = e^{-\frac{b}{2a}x}$ is the solution corresponding to the root $\lambda = -\frac{b}{2a}$. In the next slide, we'll plug in $y_1(x) = e^{-\frac{b}{2a}x}$ and $p(x) = \frac{-b}{a}$ into the Reduction of Order formula to calculate a second independent solution.

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$$y_2(x) = e^{-\frac{b}{2a}x} \int^x \frac{1}{\left(e^{-\frac{b}{2a}s}\right)^2} \exp\left[\int^s -\frac{b}{a} dt\right] ds$$

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$$y(x) = c_1 e^{-\frac{b}{2a}x} + c_2 x e^{-\frac{b}{2a}x}$$

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will be complex solutions of the characteristic equation (4).

The Main Question for Case (iii)

So how do we use the complex roots $\lambda_{\pm} = \alpha \pm i\beta$ of the characteristic equation to get functions that are solutions of the differential equations?

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We'll answer this by working out how to make sense of functions of the form

$$y(x) = e^{(\alpha + i\beta)x}$$

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The real number x on the R.H.S. of (*) is called the **real part** of z (often written $x = Re\,(z)$ and the real number y is called the **imaginary part** of z (denoted $Im\,(z)$). Thus, we specify complex numbers by prescribing their real and imaginary parts.

Polynomial Functions of a Complex Variable

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Example: Converting a polynomial to a complex function

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$$\begin{aligned} p(x) &= x^2 - 1 \\ \Rightarrow \quad p(z) &= z^2 - 1 \\ \Rightarrow \quad p(x+iy) &= (x+iy)^2 - 1 \\ &= \left(x^2 - y^2 - 1\right) + i2xy \end{aligned}$$

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We'll use the idea of Taylor expansions to make such an interpretation plausible.

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$$\frac{d}{dx}e^x = e^x \quad \Rightarrow \quad \frac{d^n}{dx^n}e^x = e^x \quad \Rightarrow \quad \left.\frac{d^n}{dx^n}e^x\right|_{x=0} = 1 \quad \text{for all } n$$

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In particular

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One finds

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&= \frac{e^{\alpha x}}{2i}\left(\cos(\beta x) + i\sin(\beta x) - \cos(\beta x) - i\sin(-\beta x)\right) \\
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We will adopt these two real-valued solutions as our fundamental solutions to the original differential equation.

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