Math 2233 - Lecture 12

Agenda

1. Summary of Constant Coefficients Case
2. Physical Interpretation: Oscillatory Systems
3. Euler-type Equations
4. Examples

Summary: Solving $ay'' + by' + cy = 0$

Method:

1. Substitute trial solution $y(x) = e^{\lambda x}$ into Constant Coefficient ODE
2. $\implies \quad a\lambda^2 + b\lambda + c = 0$
3. $\implies \quad \lambda = \lambda_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
4. Three Cases:
   (i) When $b^2 - 4ac > 0$
   - Characteristic equation has 2 distinct real roots λ_1 and λ_2
   - General solution : $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

   (ii) When $b^2 - 4ac = 0$
   - Characteristic equation has only distinct real root $\lambda = -\frac{b}{2a}$
   - General solution : $y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$

   (ii) When $b^2 - 4ac < 0$
   - Characteristic equation has 2 distinct complex roots $\alpha + i\beta$ and $\alpha - i\beta$
   - General solution : $y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$

Example 1 : $y'' - 2y' - 3y = 0$

The characteristic equation in this example is

$$\lambda^2 - 2\lambda - 3 = 0 \quad .$$

The roots of the characteristic equation are given by

$$\begin{aligned} \lambda_\pm &= \tfrac{2 \pm \sqrt{4+12}}{2} \\ &= 3, -1 \quad . \end{aligned}$$

These are distinct real roots, so the general solution is

$$y(x) = c_1 e^{3x} + c_2 e^{-x} \quad .$$

Example 2 : $y'' + 4y' + 4y = 0$

The Characteristic Equation is

$$\lambda^2 + 4\lambda + 4 = 0$$

The roots of the characteristic equation are given by

$$\begin{aligned} \lambda_\pm &= \tfrac{-4 \pm \sqrt{16-16}}{2} \\ &= -2 \quad . \end{aligned}$$

Thus, we have only one distinct root λ and the general solution is

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x} \quad .$$

Example 3 : $y'' + y' + y = 0$

The Characteristic Equation is

$$\lambda^2 + \lambda + 1 = 0$$

The roots of the characteristic equation are

$$\begin{aligned} \lambda_\pm &= \tfrac{-1 \pm \sqrt{1-4}}{2} \\ &= -\tfrac{1}{2} \pm i\tfrac{\sqrt{3}}{2} \end{aligned}$$

and so the general solution is

$$y(x) = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) \quad .$$

Physical Application: Oscillatory Systems

We've seen the general solutions to differential equations of the form

$$ay'' + by' + cy = 0$$

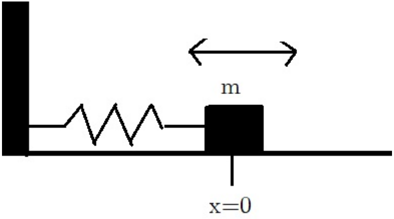
of the form

- $y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
  (when the C.Eq. has two, distinct, real roots); or
- $y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$
  (when the C.Eq. has one solution λ); or
- $y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$
  (when the C.Eq has a pair of complex roots $\alpha \pm i\beta$)

Let me spend a minute or so to connect such solutions with a simple physical application.

A Simple Oscillatory System: a mass on a spring

Here we'll consider Newton's 2nd Law of Motion, $F = ma$, applied to a simple mass-on-a-spring situation.



A box of mass m rests on a surface and is connected by a Hooke's Law type spring. We consider the situation where this mass is subject to two forces:

- When the box is moved from its equilibrium position by an displacement x, then the spring exerts a force $F_{spr} = -kx$ on the mass.
- When the box is moving it feels a frictional force due to its sliding on the surface. This force is given by $F_{fric} = -\gamma \frac{dx}{dt}$

Mass on a Spring, Cont'd

For this situation, Newton's 2nd Law says

$$F = ma \quad \Longrightarrow \quad -kx - \gamma \frac{dx}{dt} = m \frac{d^2x}{dt^2}$$

or

$$m \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + kx = 0 \qquad (1)$$

which is a 2nd order, linear, homogeneous, ODE with constant coefficients.

My aim here is to correlate the solutions to constant coefficient ODEs discussed above with various physical situations.

Frictionless Case

When $\gamma = 0$ the governing ODE is

$$m\frac{d^2x}{dt^2} + kx = 0$$

for which the characteristic equation

$$m\lambda^2 + k = 0$$

has two complex roots

$$\lambda_\pm = \pm i\sqrt{\frac{k}{m}}$$

This means we're in Case (iii) with $\alpha = 0$ and $\beta = \sqrt{\frac{k}{m}}$.

Frictionless Case, Cont'd

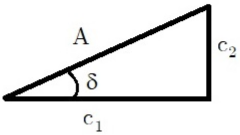
The general solution to the governing differential equation will thus be

$$\begin{aligned} x(t) &= c_1 e^{0x} \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 e^{0x} \sin\left(\sqrt{\frac{k}{m}}t\right) \\ &= c_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + c_2 \sin\left(\sqrt{\frac{k}{m}}t\right) \end{aligned}$$

To facilitate the physical interpretation of such solutions, I'll need to fiddle around with above expression a bit.

Converting the solution to a more understandable form

Consider a right triangle with sides of length c_1 and c_2.



The hypothenus then has length

$$A = \sqrt{(c_1)^2 + (c_2)^2}$$

and the adjacent angle δ is

$$\delta = \tan^{-1}\left(\frac{c_2}{c_1}\right)$$

Reversing these relations, we have

$$\begin{aligned} c_1 &= A\cos(\delta) \\ c_2 &= A\sin(\delta) \end{aligned}$$

Frictionless Case, Cont'd

So in terms of the parameters A and δ, or solution is

$$A\cos(\delta)\cos\left(\sqrt{\frac{k}{m}}t\right) + A\sin(\delta)\sin\left(\sqrt{\frac{k}{m}}t\right)$$

Employing the Trig Identity

$$\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$$

We can write

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t - \delta\right)$$

This sort of function is readily interpretable as a oscillatory motion with amplitude A, angular frequency $\sqrt{\frac{k}{m}}$ and phase shift δ.

Introducing Friction

Next, let's introduce some friction into the situation.

We'll start with just a little friction. We'll assume $\gamma^2 < 4mk$ so that the spring force continues to dominate the situation.
In this case, the characteristic equation will be

$$m\lambda^2 + \gamma\lambda + k = 0 \quad \Rightarrow \quad \lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Since we're supposing $\gamma^2 < 4mk$, the expression inside the square root will be negative, and so our roots will be complex numbers of the form

$$\lambda_\pm = \frac{-\gamma}{2m} \pm i\sqrt{\frac{4mk - \gamma^2}{4m^2}}$$

and the general solution to the governing ODE will be

$$x(t) = c_1 e^{-\frac{\gamma}{2m}t} \cos\left(\sqrt{\frac{\gamma^2 - 4mk}{4m^2}}\,t\right) + c_2 e^{-\frac{\gamma}{2m}t} \sin\left(\sqrt{\frac{\gamma^2 - 4mk}{4m^2}}\right)$$

Small Friction Case, Cont'd

Using the same trignometric trick as before we can rewrite this as

$$x(t) = Ae^{-\frac{\gamma}{2m}t}\cos\left(\sqrt{\frac{\gamma^2-4mk}{4m^2}}\,t+\delta\right)$$

$$A = \sqrt{c_1^2+c_2^2} \quad , \quad \delta = \tan^{-1}\left(\frac{c_2}{c_1}\right)$$

The cosine factor again amounts to an oscillatory type motion, but now the overall amplitude is $Ae^{-\frac{\gamma}{2m}t}$.

We thus obtain an oscillatory motion that oscillates with a particular frequency $\sqrt{\frac{\gamma^2-4mk}{4m^2}}$, but with a decaying amplitude. Exactly, what one should expect given the physical setup.

Mass on a Spring with Lots of Friction

Now let's consider another extreme.
Suppose that spring force is relatively weak compared to the frictional force. For this case, you might imagine a laboratory situation where the mass is sitting on a bed of tar or some other sticky surface.
Now when $4mk < \gamma^2$, we won't have to worry about taking the square root of a negative number and the roots of the characteristic equation are going to be a pair of distinct real numbers

$$\lambda_{\pm} = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Lots of Friction Case, Cont'd

In fact, both roots will be negative numbers : to see that the root λ_+ will be negative, just note that since $4mk < \gamma^2$,

$$\sqrt{\gamma^2 - 4mk} < \sqrt{\gamma^2} = \gamma$$

and so

$$-\gamma - \sqrt{\gamma^2 - 4mk} < 0$$

So in this case, both fundamental solutions

$$x_1(t) = e^{\lambda_+ x} \qquad \text{and} \qquad x_2(t) = e^{\lambda_- t}$$

will be decaying exponential functions. And so every solution will have the property that $x \to 0$ as $t \to \infty$.

In this situation, if you pull the mass back and then release it, there are no oscillations, rather the mass is just slowly dragged back to its equilibrium position.

Euler-type Equations

We are now going to consider how to construct solutions of another prevalent family of 2nd order, linear, homogeneous, ODEs. These will be ODEs of the form

$$ax^2y'' + bxy' + cy = 0 \quad , \tag{2}$$

where a, b and c are constants. A differential equation of this form is called an **Euler-type equation**.

Note that what characterizes an Euler type ODE is that, for each term on the left, the order of the derivative is the same as the power of x.

Solving $ax^2y'' + bxy' + cy = 0$

To solve Euler-type ODEs, we make the following **ansatz** for a trial solution:

$$y(x) = x^r \quad . \tag{3}$$

Then

$$\begin{aligned} y' &= rx^{r-1} \\ y'' &= r(r-1)x^{r-2} \end{aligned}$$

and so plugging (3) into (2) yields

$$\begin{aligned} 0 &= ax^2\left(r(r-1)x^{r-2}\right) + bx\left(rx^{r-1}\right) + cx^r \\ &= \left(ar(r-1) + br + c\right)x^r \\ &= \left(ar^2 + (b-a)r + c\right)x^r \quad . \end{aligned}$$

Solving Euler-type Equations, Cont'd

We can thus ensure that (3) is a solution of (2) by demanding

$$ar^2 + (b-a)r + c = 0$$

or

$$r = r_\pm \equiv \frac{(a-b) \pm \sqrt{(a-b)^2 - 4ac}}{2a} \quad .$$

Like that the case of second order differential equations with constant coefficients, we have three different kinds of solutions, depending on the nature of the quantity inside the square root.

Case (i): $(a-b)^2 - 4ac > 0$

In this case, the expression inside the radical is positive and we end up with two distinct real roots

$$\begin{aligned} r_+ &= \frac{a-b+\sqrt{(a-b)^2-4ac}}{2a} \\ r_- &= \frac{a-b-\sqrt{(a-b)^2-4ac}}{2a} \end{aligned}$$

and, accordingly, two linearly independent solutions

$$y_1(x) = x^{r_+} \quad , \quad y_2(x) = x^{r_-} \quad .$$

The general solution is thus

$$y(x) = c_1 x^{r_+} + c_2 x^{r_-} \quad .$$

Case (ii): $(a-b)^2 - 4ac = 0$

In this case, we only have one distinct root

$$r = \frac{a - b \pm \sqrt{(a-b)^2 - 4ac}}{2a} = \frac{a-b}{2a}$$

and so obtain only one distinct solution

$$y_1(x) = x^r = x^{\frac{a-b}{2a}} \quad .$$

A second linearly independent solution however may be found using Reduction of Order: To apply the Reduction of Order formula, we first put the differential equation in standard form so that we correctly identify the function $p(x)$

$$ax^2y''+bxy'+cy0 \quad \rightarrow \quad y''+\frac{b}{ax}y'+\frac{c}{ax^2}y = 0 \quad \Longrightarrow \quad p(x) = \frac{b}{ax}$$

Reduction of Order calculation for Case (ii)

$$\begin{aligned}
y_2(x) &= y_1(x)\int^x \frac{1}{(y_1(t))^2}\exp\left(-\int^t p(s)ds\right)dt \\
&= x^{\frac{a-b}{2a}}\int^x \frac{1}{\left(t^{\frac{a-b}{2a}}\right)^2}\exp\left(-\int^t \frac{b}{as})ds\right)dt \\
&= x^{\frac{a-b}{2a}}\int^x t^{\frac{-a+b}{a}}\exp\left(-\int^t \frac{b}{as}ds\right)dt \\
&= x^{\frac{a-b}{2a}}\int^x t^{\frac{-a+b}{a}}\exp\left(-\frac{b}{a}\ln|t|\right)dt \\
&= x^{\frac{a-b}{2a}}\int^x t^{\frac{-a+b}{a}}t^{-b/a}dt \\
&= x^{\frac{a-b}{2a}}\int^x t^{-1}t^{b/a}t^{-b/a}dt \\
&= x^{\frac{a-b}{2a}}\int^x t^{-1}dt \\
&= x^{\frac{a-b}{2a}}\ln|x| \qquad .
\end{aligned}$$

So in this case, the general solution is

$$y(x) = c_1 x^{\frac{a-b}{2a}} + c_2 x^{\frac{a-b}{2a}}\ln|x| \quad .$$

Case (iii): $(a-b)^2 - 4ac < 0$

In this case, the quantity inside the radical is negative so the roots of (the auxiliary equation are complex numbers. We set

$$\lambda = \tfrac{a-b}{2a} \qquad , \quad \mu = \frac{\sqrt{4ac-(a-b)^2}}{2a}$$

so that we can write the roots of the characteristic equation as

$$r_{\pm} = \lambda \pm i\mu$$

and the general solution as

$$y(x) = c_1 x^{\lambda + i\mu} + c_2 x^{\lambda - i\mu} \quad .$$

We now have to make sense of x raised to a complex power.

What is $x^{\lambda+i\mu}$?

We have

$$\begin{aligned} x^{\lambda+i\mu} &= \left(\exp\left(\ln|x|\right)\right)^{\lambda+i\mu} \\ &= \left(\exp\left(\ln|x|\right)\right)^{\lambda}\left(\exp\left(\ln|x|\right)\right)^{i\mu} \\ &= x^{\lambda}\left(\exp\left(i\mu\ln|x|\right)\right) \\ &= x^{\lambda}\left(\cos(\mu\ln|x|) + i\sin(\mu\ln|x|)\right) \end{aligned}$$

The real and imaginary parts of this solution will also be solutions, and, in fact, they will constitute a fundamental set of real-valued solutions to (2).

Thus, in this case the general solution will be

$$y(x) = c_1 x^{\lambda}\cos\left(\mu\ln|x|\right) + c_2 x^{\lambda}\sin\left(\mu\ln|x|\right)$$

Summary: Solving Euler-type Equations

The table below summarizes our method Euler type equations and compares that case with the constant coefficients case

	Constant Coefficients	Euler-type
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
Ansatz	$y(x) = e^{\lambda x}$	$y(x) = x^r$
Aux. Eq.	$a\lambda^2 + b\lambda + c = 0$	$ar^2 + (b-a)r + c = 0$
Case (i) 2 real roots	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$	$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$
Case (ii) 1 real root	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$	$y(x) = c_1 x^r + c_2 x^r \ln\lvert x\rvert$
Case (iii) 2 complex roots $\alpha \pm i\beta$	$y(x) = c_1 e^{\alpha x}\cos(\beta x)$ $+c_2 e^{\alpha x}\sin(\beta x)$	$y(x) = c_1 x^{\alpha}\cos(\beta \ln\lvert x\rvert)$ $+c_2 x^{\alpha}\sin(\beta \ln\lvert x\rvert)$

Example 4 : $x^2y'' - 2xy' + 2y = 0$

Substituting $y(x) = x^r$ into this differential equation yields

$$r(r-1)x^r - 2\left(rx^r\right) + 2x^r = 0$$

or

$$\left(r^2 - r - 2r + 2\right)x^r = 0$$

so we must have

$$0 = r^2 - r - 2r + 2 = r^2 - 3r + 2 = (r-2)(r-1)$$

Thus, we have $r = 2, 1$. The general solution is thus

$$y(x) = c_1x^2 + c_2x^1$$

Example 5 : $x^2y'' + 7xy' + 9y = 0$

Substituting $y(x) = x^r$ into this differential equation yields

$$r(r-1)x^r + 7\left(rx^r\right) + 9x^r = 0$$

or

$$\left(r^2 - r + 7r + 9\right)x^r = 0$$

So we must have

$$0 = r^2 - r + 7r + 9 = r^2 + 6r + 9 = (r+3)^2$$

Thus, we have only a single root of the indicial equation $r = -3$. The general solution is thus

$$y(x) = c_1 x^{-3} + c_2 \ln|x| x^{-3}$$

Example 6: $x^2y'' + xy' + 4y = 0$

Substituting $y(x) = x^r$ into this differential equation yields

$$r(r-1)x^r + (rx^r) + 4x^r = 0$$

or

$$\left(r^2 - r + r + 4\right) x^r = 0$$

so we must have

$$0 = r^2 - r + r + 4 = r^2 + 4 = (r+2i)(r-2i)$$

Thus, we have a pair of complex roots $r = 0 + 2i, 0 - 2i$.
The general solution is thus

$$\begin{aligned} y(x) &= c_1 x^0 \cos\left(2\ln|x|\right) + c_2 x^0 \sin\left(2\ln|x|\right) \\ &= c_1 \cos\left(2\ln|x|\right) + c_2 \sin\left(2\ln|x|\right) \end{aligned}$$