Math 2233 - Lecture 12

Agenda

1. Summary of Constant Coefficients Case
2. Physical Interpretation: Oscillatory Systems
3. Euler-type Equations
4. Examples

Summary: Solving $ay'' + by' + cy = 0$

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$$y(x) = c_1 e^{-\frac{1}{2}x} \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right) \quad .$$

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Let me spend a minute or so to connect such solutions with a simple physical application.

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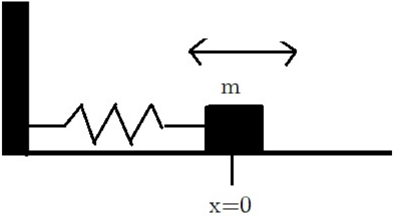
Here we'll consider Newton's 2nd Law of Motion, $F = ma$, applied to a simple mass-on-a-spring situation.



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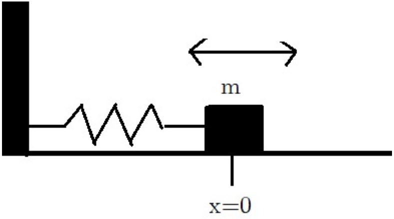

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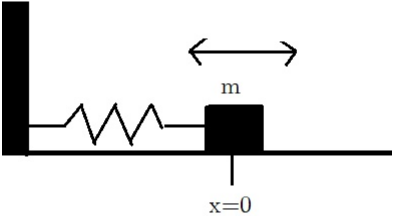

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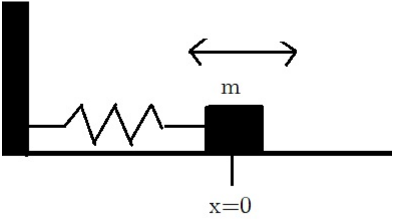

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- When the box is moved from its equilibrium position by an displacement x, then the spring exerts a force $F_{spr} = -kx$ on the mass.
- When the box is moving it feels a frictional force due to its sliding on the surface. This force is given by $F_{fric} = -\gamma \frac{dx}{dt}$

Mass on a Spring, Cont'd

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For this situation, Newton's 2nd Law says

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My aim here is to correlate the solutions to constant coefficient ODEs discussed above with various physical situations.

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This means we're in Case (iii) with $\alpha = 0$ and $\beta = \sqrt{\frac{k}{m}}$.

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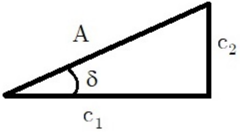
To facilitate the physical interpretation of such solutions, I'll need to fiddle around with above expression a bit.

Converting the solution to a more understandable form

Consider a right triangle with sides of length c_1 and c_2.



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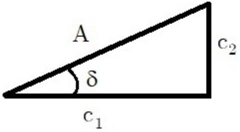

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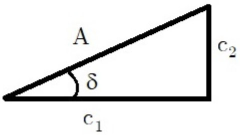

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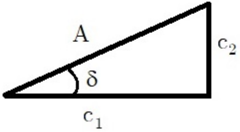

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Reversing these relations, we have

$$\begin{aligned} c_1 &= A\cos(\delta) \\ c_2 &= A\sin(\delta) \end{aligned}$$

Frictionless Case, Cont'd

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We can write

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}t - \delta\right)$$

This sort of function is readily interpretable as a oscillatory motion with amplitude A, angular frequency $\sqrt{\frac{k}{m}}$ and phase shift δ.

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and the general solution to the governing ODE will be

$$x(t) = c_1 e^{-\frac{\gamma}{2m}t} \cos\left(\sqrt{\frac{\gamma^2 - 4mk}{4m^2}}\, t\right) + c_2 e^{-\frac{\gamma}{2m}t} \sin\left(\sqrt{\frac{\gamma^2 - 4mk}{4m^2}}\right)$$

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We thus obtain an oscillatory motion that oscillates with a particular frequency $\sqrt{\frac{\gamma^2-4mk}{4m^2}}$, but with a decaying amplitude. Exactly, what one should expect given the physical setup.

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In this situation, if you pull the mass back and then release it, there are no oscillations, rather the mass is just slowly dragged back to its equilibrium position.

Euler-type Equations

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Note that what characterizes an Euler type ODE is that, for each term on the left, the order of the derivative is the same as the power of x.

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Like that the case of second order differential equations with constant coefficients, we have three different kinds of solutions, depending on the nature of the quantity inside the square root.

Case (i): $(a-b)^2 - 4ac > 0$

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In this case, the expression inside the radical is positive and we end up with two distinct real roots

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The general solution is thus

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$$ax^2y'' + bxy' + cy0 \quad \rightarrow \quad y'' + \frac{b}{ax}y' + \frac{c}{ax^2}y = 0 \quad \Longrightarrow \quad p(x) = \frac{b}{ax}$$

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$$y_2(x) \quad = \quad y_1(x) \int^x \frac{1}{(y_1(t))^2} \exp\left(-\int^t p(s)ds\right) dt$$

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so that we can write the roots of the characteristic equation as

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We now have to make sense of x raised to a complex power.

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Summary: Solving Euler-type Equations

The table below summarizes our method Euler type equations and compares that case with the constant coefficients case

	Constant Coefficients	Euler-type
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
Ansatz	$y(x) = e^{\lambda x}$	$y(x) = x^r$
Aux. Eq.	$a\lambda^2 + b\lambda + c = 0$	$ar^2 + (b-a)r + c = 0$
Case (i) 2 real roots	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$	$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$
Case (ii) 1 real root	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$	$y(x) = c_1 x^r + c_2 x^r \ln\lvert x\rvert$
Case (iii) 2 complex roots $\alpha \pm i\beta$	$y(x) = c_1 e^{\alpha x}\cos(\beta x)$ $+c_2 e^{\alpha x}\sin(\beta x)$	$y(x) = c_1 x^{\alpha}\cos(\beta \ln\lvert x\rvert)$ $+c_2 x^{\alpha}\sin(\beta \ln\lvert x\rvert)$

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Thus, we have $r = 2, 1$. The general solution is thus

$$y(x) = c_1x^2 + c_2x^1$$

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