Agenda:

1. 2nd Order Linear ODEs : Results So Far

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- 2. The Nonhomogeneous Problem
- 3. Variation of Parameters
- 4. Examples

2nd Order Linear ODEs: General Case

Standard Form

$$y'' + p(x)y' + q(x)y = g(x)$$
 (1)

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E&U Theorem: There exists 1 and only 1 solution of (1) satisfying initial conditions of the form

$$y(x_0) = y_0$$

 $y'(x_0) = y'_0$

Homogeneneous Case

Standard Form

$$y'' + p(x)y' + q(x)y = 0$$
 (0)

- Superposition Principle: If y₁(x), y₂(x) are solutions so is any function of the form y(x) = c₁y₁(x) + c₂y₂(x)
- Completeness Theorem: y₁(x), y₂(x) are solutions of (0) such that W [y₁, y₂] ≠ 0, then the general solution of (0) is given by

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

Reduction of Order: If y₁(x) is one solution of (0) then a second independent solution can be found by calculating

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x) dx\right] dx$$

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Special Cases: Constant Coefficient and Euler-type Equations

	Constant Coefficients	Euler-type
ODE	ay'' + by' + cy = 0	$ax^2y'' + bxy' + cy = 0$
Ansatz	$y(x) = e^{\lambda x}$	$y(x) = x^r$
Aux. Eq.	$a\lambda^2 + b\lambda + c = 0$	$ar^2 + (b-a)r + c = 0$
Case (i) 2 real roots	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$	$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$
Case (ii) 1 real root	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$	$y(x) = c_1 x^r + c_2 x^r \ln x $
Case (iii) 2 complex roots $\alpha \pm i\beta$	$y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$	$y(x) = c_1 x^{\alpha} \cos(\beta \ln x) + c_2 x^{\alpha} \sin(\beta \ln x)$

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The Nonhomogeneous Problem

We now consider differential equations of the form

$$y'' + p(x)y' + q(x)y = g(x)$$
 (1)

where $g(x) \neq 0$.

Some things that we'd obviously like to know are

- how to construct solutions; and
- how to know if we have all the solutions.

To this end, it certainly would be nice to have something like the Superposition Principle at our disposal.

However, for non-homogeneous linear differential equations the Superposition Principle is not applicable.

Failure of Superposition Principle in Nonhomogeneous Case

Suppose $Y_1(x)$ and $Y_2(x)$ are solutions of (1). If the Superposition Principle were valid, then $Y(x) = c_1 Y_1(x) + c_2 Y_2(x)$ would also be a solution. But for this Y(x)

$$Y'' + p(x)Y' + q(x)Y = c_1Y''_1 + c_2Y''_2 + p(x)(c_1Y'_1 + c_2Y'_2) +q(x)(c_1Y_1 + c_2Y_2) = c_1(Y''_1 + p(x)Y'_1 + q(x)Y_1) +c_2(Y''_2 + p(x)Y'_2 + q(x)Y_2) = c_1g(x) + c_2g(x) = (c_1 + c_2)g(x) \neq g(x)$$

An Alternative to the Superposition Principle

Thus, if $Y_1(x)$ and $Y_2(x)$ satisfy (1) then a linear combination of Y_1 and Y_2 **does not** satisfy

$$Y'' + p(x)Y' + q(x)Y = g(x)$$
 (1)

Rather, the calculation on the preceding slide tells us that a linear combination $\tilde{Y}(x) = c_1 Y_1(x) + c_2 Y_2(x)$ satisfies a different ODE

$$\tilde{Y}'' + p(x)\tilde{Y}' + q(x)\tilde{Y} = (c_1 + c_2)g(x)$$
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Nevertheless, Eq. (*) provides us with an alternative path towards the general solution of (1)

Let $Y_1(x)$ and $Y_2(x)$ be any two solutions of (1) and set $\Delta Y(x)$ defined as the difference of $Y_1(x)$ and $Y_2(x)$:

$$\Delta Y(x) \equiv Y_1(x) - Y_2(x) \,.$$

Applying the preceding calculation with $c_1 = 1$ and $c_2 = -1$ we see that Y(x) obeys

$$ilde{Y}''+p(x) ilde{Y}'+q(x) ilde{Y}=(1-1)g(x)=0$$

Thus, the difference ΔY of any two solutions of (1) must be a solution of the corresponding homogeneous equation.

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An Alternative to the Superposition Principle, Cont'd

Now let's assume that we've already solved the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$
 (0)

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More precisely, suppose we have found two indep. solutions, $y_1(x)$ and $y_2(x)$ of (3) and so, since

$$\Delta Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

also satisfies (0), we must have

$$Y_1(x) - Y_2(x) \equiv \Delta Y(x) = c_1 y_1(x) + c_2 y_2(x)$$

(N.B. Y_1 , Y_2 are solutions of (1), while y_1 , y_2 are solutions of (0))

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An Alternative to the Superposition Principle, Cont'd

$$Y_1(x) = Y_2(x) + c_1 y_1(x) + c_2 y_2(x)$$

This last equation says that if we know 1 solution, $Y_2(x)$, of

$$Y'' + p(x)Y' + q(x)Y = g(x)$$
(1)

and 2 independent solutions of

$$y'' + p(x)y' + q(x)y = 0$$
 (0)

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then any other solution, $Y_1(x)$, of (1) can be expressed as

$$Y_1(x) = Y_2(x) + c_1 y_1(x) + c_2 y_2(x)$$

Form of the General Solution for the Nonhomogeneous Case

The following theorem summarizes this discussion and provides the foundation upon which we can construct solutions of nonhomogeneous second order linear ODEs.

Theorem

Suppose $Y_p(x)$ is a particular solution of the nonhomogeneous linear ODE

$$Y'' + p(x)Y' + q(x)Y = g(x)$$
 (1)

and that $y_1(x)$, $y_2(x)$ are two independent solutions of the corresponding homogeneous linear ODE

$$y'' + p(x)y' + q(x)y = 0$$
 (0)

then every solution of (1) can be expressed as

$$Y(x) = Y_{p}(x) + c_{1}y_{1}(x) + c_{2}y_{2}(x)$$

Method for Solving Nonhomogeneous Case

Thus, to determine the general solution of a non-homogeneous linear equation

$$Y'' + p(x)Y' + q(x)Y = g(x)$$
(1)

we can procede in three steps.

1. Determine two independent solutions, $y_1(x), y_2(x)$ of the corresponding homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0$$
 (0)

- 2. Find a particular solution $Y_p(x)$ of the nonhomogeneous differential equation (1).
- 3. Construct the general solution of (1) by setting

$$Y(x) = Y_p(x) + c_1 y_1(x) + c_2 y_2(x)$$
(4)

Example

Given that one solution of

$$Y'' + 3Y' + 2Y = e^{-x}$$
(5)

is

$$Y(x) = xe^{-x}$$

find the general solution of (5).

We first identify the function $Y_p(x)$ in the theorem statement with our given solution xe^{-x} .

$$Y_p(x) = xe^{-x}$$

Next, we need to find 2 independent solutions of

$$y'' + 3y' + 2y = 0 (6)$$

Note that (6) is a Constant Coefficients type ODE with Characteristic Equation

$$\lambda^2 + 3\lambda + 2 = 0$$

which factors as

$$(\lambda + 2)(\lambda + 1) = 0$$

Thus, we have two real roots $\lambda = -2, -1$ and hence two linearly solutions of (6)

$$y_1(x) = e^{-2x}$$

 $y_2(x) = e^{-x}$

We now have all the ingredients we need to write down the general solution of (5):

$$Y(x) = Y_p(x) + c_1y_1(x) + c_2y_2(x)$$

= $xe^{-x} + c_1e^{-2x} + c_2e^{-x}$

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How to find $Y_p(x)$?

We now turn to the problem of finding that first solution Y_p of a nonhomogeneous linear ODE.

Consider the differential equation

$$Y'' + p(x)Y' + q(x)Y = g(x)$$
 (1)

Suppose $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of the corresponding homogeneous problem

$$y'' + p(x)y' + q(x)y = 0$$
 (0)

We will seek to find two functions $u_1(x)$ and $u_2(x)$ such that

$$Y_{p}(x) = u_{1}(x)y_{1}(x) + u_{2}(x)y_{2}(x)$$
(7)

is a solution of (1).

To determine the two functions u_1 and u_2 uniquely, we need to impose two independent conditions on the unknwon functions u_1 and u_2

First, we shall require

$$Y_{p}(x) = u_{1}(x)y_{1}(x) + u_{2}(x)y_{2}(x)$$
(7)

to be a solution of (1); Secondly, we shall require

$$u_1'y_1 + u_2'y_2 = 0 \quad . \tag{8}$$

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(This auxiliary condition is imposed not only because we need a second equation, but also to simplify the calculation of derivative terms)

Differentiating (7) yields

$$Y'_{\rho} = u'_{1}y_{1} + u_{1}y'_{1} + u'_{2}y_{2} + u_{2}y'_{2}$$
(9)

which because of (8) becomes

$$Y'_{p} = u_1 y'_1 + u_2 y'_2 \quad . \tag{10}$$

Differentiating again yields

$$Y''_{p} = u'_{1}y'_{1} + u_{1}y''_{1} + u'_{2}y'_{2} + u_{2}y''_{2} \quad . \tag{11}$$

We now plug (7), (10), and (11) into the original differential equation (1).

$$g(x) = (u'_1y'_1 + u_1y''_1 + u'_2y'_2 + u_2y''_1) +p(x)(u_1y'_1 + u_2y'_2) +q(x)(u_1y_1 + u_2y_2) = u'_1y'_1 + u'_2y'_2 +u_1(y''_1 + p(x)y'_1 + q(x)y_1) +u_2(y''_2 + p(x)y'_2 + q(x)y_2)$$

The last two terms vanish since y_1 and y_2 are solutions of (8). We thus have

$$u_1'y_1' + u_2'y_2' = g \tag{12}$$

along with the auxiliary equation

$$u_1'y_1 + u_2'y_2 = 0 (10)$$

We can solve the two linear equations, (12) and (10), for u'_1 and u'_2 .

Solving 2 Linear Equations in 2 Unknowns

However, rather than explicitly carry out the algebraic solution of equations (10) and (12), we'll use the following theorem from Linear Algebra:

Theorem

Let

$$Ax + By = e$$
$$Cx + Dy = f$$

be a pair of independent linear equations in two unknowns x and y. Then the solution of this system is given by

$$x = \frac{eD - Bf}{AD - BC}$$
$$y = \frac{Af - eC}{AD - BC}$$

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Thus, in the situation at hand, regarding (12a) and (12b) as a pair of linear equations for u'_1 and u'_2 , we have

$$\begin{array}{rcl} u_1' & = & \frac{-y_2g}{y_1y_2'-y_1'y_2} = \frac{-y_2g}{W[y_1,y_2]} \\ u_2' & = & \frac{y_1g}{y_1y_2'-y_1'y_2} = \frac{y_1g}{W[y_1,y_2]} \end{array}$$

(Note that division by $W(y_1, y_2)$ causes no problems since y_1 and y_2 were chosen such that $W(y_1, y_2) \neq 0$.) Hence

$$u_{1}(x) = -\int^{x} \frac{y_{2}(t)g(t)}{W[y_{1},y_{2}](t)}dt$$

$$u_{2}(x) = \int^{x} \frac{y_{1}(t)g(t)}{W[y_{1},y_{2}](t)}dx'$$

and so

$$Y_{p}(x) = -y_{1}(x) \int^{x} \frac{y_{2}(t)g(t)}{W[y_{1}, y_{2}](t)} dt + y_{2}(x) \int^{x} \frac{y_{1}(t)g(t)}{W[y_{1}, y_{2}](t)} dt$$

is a particular solution of (1).

Summary: Finding the 1st Solution Y_p

We have just proved:

Theorem (Variation of Parameters Formula) Suppose $y_1(x)$. $y_2(x)$ are two independent solutions of

$$y'' + p(x)y' + q(x)y = 0$$

Then a particular solution $Y_p(x)$ of

$$Y'' + p(x)Y' + q(x)Y = g(x)$$

can be calculated as

$$Y_{p}(x) = -y_{1}(x) \int^{x} \frac{y_{2}(t)g(t)}{W[y_{1}, y_{2}](t)} dt + y_{2}(x) \int^{x} \frac{y_{1}(t)g(t)}{W[y_{1}, y_{2}](t)} dt$$

Example

Find the general solution of

$$y'' - y' - 2y = 2e^{-x}$$
(13)

using the method of Variation of Parameters. Well, the corresponding homogeneous problem is

$$y'' - y' - 2y = 0 \quad . \tag{14}$$

This is a second order linear equation with constant coefficients whose characteristic equation is

$$\lambda^2 - \lambda - 2 = 0$$

The characteristic equation has two distinct real roots

$$\lambda = -1, 2$$

and so the functions

$$y_1(x) = e^{-x}$$

 $y_2(x) = e^{2x}$

form a fundamental set of solutions to (14).

To find a particular solution to (14) we employ the Variation of Parameters Theorem.

$$Y_{p}(x) = -y_{1}(x) \int^{x} \frac{y_{2}(t)g(t)}{W[y_{1}, y_{2}](t)} dt + y_{2}(x) \int^{x} \frac{y_{1}(t)g(t)}{W[y_{1}, y_{2}](t)} dt$$

Now, in the problem at hand,

$$g(x) = 2e^{-x}$$

and

$$W[y_1, y_2](x) = (e^{-x})(2e^{2x}) - (-e^{-x})(e^{2x}) = 3e^x$$
,

So

$$Y_{\rho}(x) = -y_{1}(x) \int^{x} \frac{y_{2}(t)g(t)}{W[y_{1},y_{2}](t)} dt + y_{2}(x) \int^{x} \frac{y_{1}(t)g(t)}{W[y_{1},y_{2}](t)} dt$$

$$= -e^{-x} \int^{x} \frac{e^{2t}(2e^{-t})}{3e^{t}} dt + e^{2x} \int^{x} \frac{e^{-t}(2e^{-t})}{3e^{t}} dt$$

$$= -e^{-x} \int^{x} \frac{2}{3} dt + e^{2x} \int^{x} \frac{2}{3} e^{-3t} dt$$

$$= -\frac{2}{3} x e^{-x} - \frac{2}{9} e^{-x}$$

The general solution of (13) is thus

$$Y(x) = Y_p(x) + c_1y_1(x) + c_2(x) = \left(-\frac{2}{3}xe^{-x} - \frac{2}{9}e^{-x}\right) + c_1e^{-x} + c_2e^{2x} = -\frac{2}{3}xe^{-x} + \left(c_1 - \frac{2}{9}\right)e^{-x} + c_2e^{2x} = -\frac{2}{3}xe^{-x} + C_1e^{-x} + C_2e^{2x}$$

where we have absorbed the constant factor $-\frac{2}{9}$ in the 3rd line into the arbitrary parameter C_1 .

Example

$$x^2y'' - 5xy' + 9y = x^3$$

First we solve the corresponding homogeneous equation

$$x^2y''-5xy'+9y=0$$

Note that the homogeneous ODE is of Euler type. Substituting $y(x) = x^r$ into this equation yields

$$0 = r(r-1)x^{r} - 5rx^{r} + 9x^{r} = (r^{2} - 6r + 9)x^{r}$$

So we must have

$$0 = r^2 - 6r + 9 = (r - 3)^2$$

We thus have a single real root of the indicial equation; r = 3. The corresponding linearly independent real-valued solutions of the original Euler-type differential equation are

$$y_1(x) = x^3$$

 $y_2(x) = x^3 \ln |x|$

The Wronskian of $y_1(x)$ and $y_2(x)$ is

$$W[y_1, y_2](x) = x^3 \left(3x^2 \ln |x| + x^3 \left(\frac{1}{x}\right) - (3x^2) x^3 \ln |x| \right)$$

= 3x⁵ ln |x| + x⁵ - 3x⁵ ln |x|
= x⁵

To identify the function g(x) in the Variation of Parameters formula we must first cast the original non-homogeneous equation into its Standard Form:

$$y'' - \frac{5}{x}y' + \frac{9}{x^2}y = x.$$

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Hence, g(x) = x.

Thus we have

$$Y_{p}(x) = -y_{1}(x) \int^{x} \frac{y_{2}(s)g(s)}{W[y_{1}, y_{2}](s)} ds + y_{2}(x) \int^{x} \frac{y_{1}(s)g(s)}{W[y_{1}, y_{2}](s)} ds$$

$$= -x^{3} \int^{x} \frac{(s^{3} \ln |s|) s}{s^{5}} ds + x^{3} \ln |x| \int^{x} \frac{(s^{3}) s}{s^{5}} ds$$

$$= -x^{3} \int^{x} \ln |s| (s^{-1}ds) + x^{3} \ln |x| \int^{x} \frac{1}{s} ds$$

$$= -x^{3} \int^{u=\ln |x|} u du + x^{3} \ln |x| (\ln |x|)$$

$$= -x^{3} \left(\frac{1}{2} (\ln |x|)^{2}\right) + x^{3} (\ln |x|)^{2}$$

$$= \frac{1}{2}x^{3} (\ln |x|)^{2}$$

Thus the general solution is

$$Y(x) = \frac{1}{2}x^{3} (\ln|x|)^{2} + c_{1}x^{3} + c_{2}x^{3} \ln|x|$$