Math 2233 - Lecture 13

Agenda:

1. 2nd Order Linear ODEs : Results So Far
2. The Nonhomogeneous Problem
3. Variation of Parameters
4. Examples

2nd Order Linear ODEs: General Case

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- Standard Form

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$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)dx\right] dx$$

Special Cases: Constant Coefficient and Euler-type Equations

	Constant Coefficients	Euler-type
ODE	$ay'' + by' + cy = 0$	$ax^2y'' + bxy' + cy = 0$
Ansatz	$y(x) = e^{\lambda x}$	$y(x) = x^r$
Aux. Eq.	$a\lambda^2 + b\lambda + c = 0$	$ar^2 + (b - a)r + c = 0$
Case (i) 2 real roots	$y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$	$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$
Case (ii) 1 real root	$y(x) = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$	$y(x) = c_1 x^r + c_2 x^r \ln\lvert x\rvert$
Case (iii) 2 complex roots $\alpha \pm i\beta$	$y(x) = c_1 e^{\alpha x}\cos(\beta x)$ $+c_2 e^{\alpha x}\sin(\beta x)$	$y(x) = c_1 x^\alpha \cos(\beta \ln\lvert x\rvert)$ $+c_2 x^\alpha \sin(\beta \ln\lvert x\rvert)$

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To this end, it certainly would be nice to have something like the Superposition Principle at our disposal.
However, for non-homogeneous linear differential equations the Superposition Principle is not applicable.

Failure of Superposition Principle in Nonhomogeneous Case

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&= c_1\left(Y_1'' + p(x)Y_1' + q(x)\,Y_1\right) \\
&\quad + c_2\left(Y_2'' + p(x)Y_2' + q(x)Y_2\right) \\
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Thus, if $Y_1(x)$ and $Y_2(x)$ satisfy (1) then a linear combination of Y_1 and Y_2 **does not** satisfy

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Nevertheless, Eq. (*) provides us with an alternative path towards the general solution of (1)

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Thus, the difference ΔY of any two solutions of (1) must be a solution of the corresponding homogeneous equation.

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then every solution of (1) can be expressed as

$$\boxed{Y(x) = Y_p(x) + c_1 y_1(x) + c_2 y_2(x)}$$

Method for Solving Nonhomogeneous Case

Thus, to determine the general solution of a non-homogeneous linear equation

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we can procede in three steps.

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Thus, to determine the general solution of a non-homogeneous linear equation

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$$y'' + 3y' + 2y = 0 \tag{6}$$

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We will seek to find two functions $u_1(x)$ and $u_2(x)$ such that

$$Y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{7}$$

is a solution of (1).

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We can solve the two linear equations, (12) and (10), for u_1' and u_2'.

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$$\begin{aligned} x &= \frac{eD - Bf}{AD - BC} \\ y &= \frac{Af - eC}{AD - BC} \end{aligned}$$

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$$W[y_1,y_2](x) = \left(e^{-x}\right)\left(2e^{2x}\right) - \left(-e^{-x}\right)\left(e^{2x}\right) = 3e^x \quad ,$$

So

$$\begin{aligned} Y_p(x) &= -y_1(x)\int^x \tfrac{y_2(t)g(t)}{W[y_1,y_2](t)}dt \quad + \quad y_2(x)\int^x \tfrac{y_1(t)g(t)}{W[y_1,y_2](t)}dt \\ &= -e^{-x}\int^x \tfrac{e^{2t}\left(2e^{-t}\right)}{3e^t}dt \quad + \quad e^{2x}\int^x \tfrac{e^{-t}\left(2e^{-t}\right)}{3e^t}dt \\ &= -e^{-x}\int^x \tfrac{2}{3}dt \quad + \quad e^{2x}\int^x \tfrac{2}{3}e^{-3t}dt \\ &= -\tfrac{2}{3}xe^{-x} - \tfrac{2}{9}e^{-x} \end{aligned}$$

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where we have absorbed the constant factor $-\frac{2}{9}$ in the 3rd line into the arbitrary parameter C_1.

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