Math 2233 - Lecture 14

Agenda:

1. Summary: Solving 2nd Order Linear ODEs
2. Examples
3. The Laplace Transform
4. The Laplace Transform Method for Solving ODEs

To solve

$$y'' + p(x)\, y' + q(x)\, y = g(x) \tag{1}$$

1. Find at least 1 independent solution $y_1(x)$ of

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{0}$$

2. If only 1 solution is found in Step 1, calculate a 2nd independent solution of (0)

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\, dx\right] dx \tag{2}$$

3. The general solution of (0) is now

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

4. Calculate 1st solution Y_p of (1)

$$Y_p(x) = -y_1(x) \int \frac{y_2(x)\, g(x)}{W[y_1, y_2](x)} dx + y_2(x) \int \frac{y_1(x)\, g(x)}{W[y_1, y_2](x)} dx \tag{4}$$

5. The general solution of (1) is now

$$Y(x) = Y_p(x) + c_1 y_1(x) + c_2 y_2(x) \tag{5}$$

Example

Solve the following initial value problem

$$\begin{aligned} y'' - 3y' + 2y &= 10 \\ y(0) &= 5 \\ y'(0) &= -1 \end{aligned}$$

Step 1: We'll first solve the corresponding homogeneous equation

$$y'' - 3y' + 2y = 0$$

which is second order linear with constant coefficients. Its characteristic equation is

$$0 = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$$

and so we have two real roots $\lambda = 2, 1$ and two linearly independent solutions

$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= e^{x} \end{aligned}$$

Step 2: We will now use the Variation of Parameters to determine a particular solution $Y_p(x)$ of the original nonhomogeneous equation. First note that

$$\begin{aligned} g(x) &= 10 \\ W[y_1, y_2](x) &= \left(e^{2x}\right)\left(e^{x}\right) - \left(2e^{2x}\right)\left(e^{x}\right) = -e^{3x} \end{aligned}$$

and so

$$\begin{aligned} y_p(x) &= -y_1(x)\int^x \frac{y_2(s)g(s)}{W[y_1, y_2](s)}ds + y_2(x)\int^x \frac{y_1(s)g(s)}{W[y_1,y_2](s)}ds \\ &= -e^{2x}\int^x \frac{e^s(10)}{-e^{3s}}ds + e^x\int^x \frac{e^{2s}(10)}{-e^{3s}}ds \\ &= +10e^{2x}\int^x e^{-2s}ds - 10e^x\int^x e^{-s}ds \\ &= 10e^{2x}\left(-\frac{1}{2}e^{-2x}\right) - 10e^x\left(-e^{-x}\right) \\ &= -5 + 10 \\ &= 5 \end{aligned}$$

Thus the general solution is

$$\begin{aligned} y(x) &= y_p(x) + c_1 y_1(x) + c_2 y_2(x) \\ &= 5 + c_1 e^{2x} + c_2 e^{x} \end{aligned}$$

Step 3: Use the initial conditions to find appropriate values for c_1 and c_2

We have

$$\begin{aligned} 5 &= y(0) = 5 + c_1 + c_2 \quad \Rightarrow \quad c_1 + c_2 = 0 \\ -1 &= y'(0) = 2c_1 + c_2 \quad \Rightarrow \quad 2c_1 + c_2 = -1 \end{aligned}$$

and so

$$\begin{aligned} c_1 &= -1 \\ c_2 &= 1 \end{aligned}$$

The solution to the initial value problem is thus

$$y(x) = 5 + e^{2x} - e^{x}$$

The Laplace Transform

Definition

The **Laplace transform** of a function $f(x)$ is

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x)\, dx \quad . \tag{6}$$

We note that in the formula (6), s is the new variable upon which the Laplace transform $\mathcal{L}[f]$ depends.

Example 1

$$
\begin{aligned}
\mathcal{L}\left[1\right](s) &\equiv \int_0^{\infty} (1)\, e^{-sx} dx \\
&= -\frac{1}{s} e^{-sx} \Big|_{x=0}^{x=\infty} \\
&= 0 - \left(-\frac{1}{s}\right) \\
&= \frac{1}{s}
\end{aligned}
$$

Example 2: $\mathcal{L}[x^n]$

$$\mathcal{L}[x^n](s) \equiv \int_0^\infty x^n e^{-sx}\,dx$$

Integration by Parts: $\int_a^b u\,dv = uv|_a^b - \int_a^b v\,du$

$$\begin{array}{ccc} u = x^n & dv = e^{-sx}dx & \\ \downarrow \frac{d}{dx} & \downarrow \int & \backslash \\ du = nx^{n-1}dx & v = -\frac{1}{s}e^{-sx} & \end{array}$$

$$\begin{aligned} \Rightarrow \quad \mathcal{L}[x^n](s) &= (x^n)\left(-\frac{1}{s}e^{-sx}\right)\Big|_{x=0}^{x=\infty} - \int_0^\infty nx^{n-1}\left(-\frac{1}{s}e^{-sx}\right)dx \\ &= 0 - 0 + \frac{n}{s}\int x^{n-1}e^{-sx}dx \\ &= \frac{n}{s}\mathcal{L}\left[x^{n-1}\right] \end{aligned}$$

Thus,

$$
\begin{aligned}
\mathcal{L}\left[x^n\right] &= \frac{n}{s}\mathcal{L}\left[x^{n-1}\right] \\
&= \frac{n}{s}\frac{n-1}{s}\mathcal{L}\left[x^{n-2}\right] \\
&= \frac{n}{s}\frac{n-1}{s}\cdots\frac{1}{s}\mathcal{L}\left[x^{0}\right] \\
&= \frac{n}{s}\frac{n-1}{s}\cdots\frac{1}{s}\mathcal{L}\left[1\right] \\
&= \frac{n}{s}\frac{n-1}{s}\cdots\frac{1}{s}\frac{1}{s} \\
&= \frac{n!}{s^{n+1}}
\end{aligned}
$$

Example 3

If $f(x) = e^{bx}$, then

$$\begin{aligned} \mathcal{L}[f] &= \int_0^\infty e^{bt} e^{-st}\, dt \\ &= \int_0^\infty e^{(b-s)t}\, dt \\ &= \frac{1}{b-s} e^{(b-s)t} \Big|_0^\infty \\ &= \frac{1}{s-b} \quad (\text{if } s > b) \end{aligned}$$

(If $s \leq b$ then the integral does not converge.)

Example 4

If

$$f(x) = \sin(ax)$$

then, integrating by twice by parts,

$$\begin{aligned}
\mathcal{L}[f](s) &= \int_0^\infty \sin(ax) e^{-sx}\, dx \\
&= \lim_{N\to\infty} \left(e^{-sx} \tfrac{1}{a} \cos(ax) \right)\Big|_0^N + \tfrac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \\
&= \tfrac{1}{a} + \tfrac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \\
&= \tfrac{1}{a} + \lim_{N\to\infty} \tfrac{s}{a} \left(-\tfrac{1}{a} e^{-sx} \sin(ax) \right)\Big|_0^N - \tfrac{s^2}{a^2} \int_0^\infty e^{-sx} \sin(ax)\, dx \\
&= \tfrac{1}{a} + 0 - \tfrac{s^2}{a^2} L[f](s) \quad ,
\end{aligned}$$

we find

$$\mathcal{L}[f](s) = \frac{a}{1 + \frac{s^2}{a^2}} = \frac{a}{a^2 + s^2} \quad .$$

(If $s \le 0$, the integral on the first line does not converge, so $\mathcal{L}[f](s)$ is only defined for $s > 0$.)

A Table of Basic Laplace Transforms

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\mathcal{L}\left[t^n e^{at}\right] = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2 + b^2}$$

$$\mathcal{L}\left[e^{at}\cos(bt)\right] = \frac{s-a}{(s-a)^2 + b^2}$$

$$\mathcal{L}\left[e^{at}\sinh(bt)\right] = \frac{b}{(s-a)^2 - b^2}$$

$$\mathcal{L}\left[e^{at}\cosh(bt)\right] = \frac{s-a}{(s-a)^2 - b^2}$$

Formal Properties of the Laplace Transform

Theorem

(i) $\mathcal{L}\left[c_1 f_1 + c_2 f_2\right] = c_1 \mathcal{L}\left[f_1\right] + c_2 \mathcal{L}\left[f_2\right]$

(ii) $\mathcal{L}\left[\frac{df}{dx}\right] = s\mathcal{L}\left[f\right] - f\left(0\right)$

(iii) $\mathcal{L}\left[\frac{d^2 f}{dx^2}\right] = s^2\mathcal{L}\left[f\right] - sf\left(0\right) - f'\left(0\right)$

Proof of (i) : $\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$

This follows from the linearity property integration:

$$
\begin{aligned}
\mathcal{L}[c_1 f_1 + c_2 f_2](s) &= \int_0^\infty (c_1 f_1(x) + c_2 f_2(x))\, e^{-sx} dx \\
&= c_1 \int^x f_1(x) e^{-sx} dx \quad + \quad c_2 \int^x f_2(x) e^{-sx} dx \\
&= c_1 \mathcal{L}[f_1](s) + c_2 \mathcal{L}[f_2](s)
\end{aligned}
$$

Proof of (ii) : $\mathcal{L}\left[\frac{df}{dx}\right] = s\mathcal{L}\left[f\right] - f\left(0\right)$

Integrating by parts one finds

$$
\begin{aligned}
\mathcal{L}[f'](s) &= \int_0^\infty e^{-st} f'(t) dt \\
&= e^{-st} f(t)|_0^\infty - \int_0^\infty \left(-se^{-st}\right) f(t) dt \\
&= 0 - f(0) + s \int_0^\infty e^{-st} f(t) dt \\
&= s\mathcal{L}[f] - f(0) \quad .
\end{aligned}
$$

Proof of (iii) : $\mathcal{L}\left[\frac{d^2 f}{dx^2}\right] = s^2\mathcal{L}[f] - sf(0) - f'(0)$

To prove (iii), we can use property (ii)

$$\begin{aligned}\mathcal{L}\left[\frac{d^2 f}{dx^2}\right] &= s\mathcal{L}\left[\frac{df}{dx}\right] - \frac{df}{dx}(0) \\ &= s\left(s\mathcal{L}[f] - f(0)\right) - \frac{df}{dx}(0) \\ &= s^2\mathcal{L}[f] - sf(0) - \frac{df}{dx}(0)\end{aligned}$$

Application of Laplace Transforms to Initial Value Problems

Consider the following initial value problem.

$$\begin{aligned} y'' - y' - 2y &= 0 \\ y(0) &= 2 \\ y'(0) &= 4 \end{aligned} \tag{7}$$

We could treat this as a Constant Coefficient type ODE

We will develop here another method based on the Laplace transform.

Let's take the Laplace transform of the ODE:

$$\mathcal{L}\left[y'' - y' - 2y\right] = \mathcal{L}[0]$$

or

$$\mathcal{L}\left[y''\right] - \mathcal{L}\left[y'\right] - 2\mathcal{L}[y] = 0$$

We have

$$\begin{aligned} \mathcal{L}\left[y''\right] &= s^2\mathcal{L}[y] - sy(0) - y'(0) \\ &= s^2\mathcal{L}[y] - 2s - 4 \end{aligned}$$

$$\begin{aligned} \mathcal{L}\left[y'\right] &= s\mathcal{L}[y] - y(0) \\ &= s\mathcal{L}[y] - 2 \end{aligned}$$

Thus

$$\left(s^2\mathcal{L}[y] - 2s - 4\right) - \left(s\mathcal{L}[y] - 2\right) - 2\mathcal{L}[y] = 0$$

or

$$\left(s^2 - s - 2y\right)\mathcal{L}\left[y\right] - 2s - 2 = 0$$

or

$$\begin{aligned}
\mathcal{L}\left[y\right] &= \frac{2s+2}{s^2 - s - 2y} \\
&= \frac{2s+2}{\left(s+1\right)\left(s-2\right)} \\
&= 2\frac{1}{s-2} \\
&= 2\mathcal{L}\left[e^{2x}\right] \\
&= \mathcal{L}\left[2e^{2x}\right]
\end{aligned}$$

We conclude

$$y\left(x\right) = 2e^{2x}$$

Solving Initial Value Problems Using the Laplace Transform

1. Take the Laplace transform of both sides of the ODE using the identities

$$\begin{aligned} \mathcal{L}[y'](s) &= s\mathcal{L}[y] - y(0) \\ \mathcal{L}[y''](s) &= s^2\mathcal{L}[y] - sy(0) - y'(0) \end{aligned}$$

for the derivative terms.

2. Use the specified initial values for $y(0)$ and $y'(0)$
3. Solve the resulting algebraic equation in order to express $\mathcal{L}[y]$ as an explicit function of s.
4. Try to identify a function $f(x)$ such that $\mathcal{L}[f](s)$ is the function $\mathcal{L}[y]$ of s found in Step 3.
5. The solution of the differential equation will be the function $f(x)$ determined in Step 4.

In what follows, we shall be concentrating on Step 4.

Inverting the Laplace Transforms of Rational Functions

After the first three steps of the procedure outlined on the preceding page, we arrive at an equation that expresses the Laplace transform $\mathcal{L}[y]$ of our solution as a function of s, the Laplace transform variable. Typically, this equation will look like

$$\mathcal{L}[y] = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are polynomials.

There will be basic three cases to consider; depending on the nature of denominator $Q(x)$.

1. $Q(s)$ can be completely factored. In this case, we'll use Partial Fractions expansions to invert the Laplace transform.
2. $Q(x)$ is of the form $(s-a)^2 + b^2$ (the denominator Q is a sum of squares)
3. $Q(x)$ is of the form $(s-a)^2 - b^2$ (the denominator Q is a difference of squares)

Digression: Review of Partial Fractions

Partial fraction expansions are a very useful tool for figuring out inverse Laplace transforms. A simple way to understand Partial Fractions expansions is that they reverse the algebra that goes into putting a sum of rational functions over a common denominator. For example,

$$\frac{2}{s+1} + \frac{3}{s-2} = \frac{2\,(s-2) + 3\,(s+1)}{(s+1)\,(s-2)} = \frac{5s-1}{(s+1)\,(s-2)}$$

or

$$\frac{5s-1}{(s+1)\,(s-2)} = \frac{2}{s+1} + \frac{3}{s-2}$$

In the latter equation, the right hand side is the Partial Fractions Expansion of $\frac{5s-1}{(s+1)(s-2)}$.

The following theorem prescribes the form of a Partial Fractions Expansion of common family of rational functions.

Theorem

Suppose

$$Q(s) = \prod_{i=1}^{k} (s - a_i)^{m_i} \equiv (s - a_1)^{m_1} (s - a_2)^{m_2} \cdots (s - a_k)^{m_k}$$

and $P(s)$ *is a polynomial such that* $\deg(P) < \deg(Q)$.
Then there exists numbers $a_{ij}, i = 1..k, j = 1, \ldots, m_i$, *such that*

$$\frac{P(s)}{Q(s)} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{a_{ij}}{(s - a_i)^j}$$

Here is an example as to how this theorem is applied.

$$\frac{s^2 - 3s + 1}{(s-3)(s+2)^3} = \frac{a_{11}}{s-3} + \frac{a_{21}}{s+2} + \frac{a_{22}}{(s+2)^2} + \frac{a_{23}}{(s+2)^3}$$

for some particular numbers a_{11}, a_{21}, a_{22} and a_{23}
(I'll discuss below how to find the correct values for these numbers.)
Rather than introducing indexed symbols a_{ij}. One typically just uses different letters to represent the numbers in the numerators on the right; e.g., as in

$$\frac{s^2 - 3s + 1}{(s-3)(s+2)^3} = \frac{A}{s-3} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3} \qquad (*)$$

Also, rather than relying on the formula in the theorem, one usually constructs the Partial Fractions Expansion by simply adding up the contributions for each factor $\frac{1}{(s-a)^m}$ in $F(s)$ in the denominator $Q(x)$;

- a factor $(s-a)$ in the denominator leads to a term of the form $\frac{A}{s-z}$ in the partial fractions expansion
- a factor $(s-a)^2$ in the denominator leads to two terms, $\frac{A}{s-a} + \frac{B}{(s-a)^2}$ in the partial fractions expansion
- a factor $(s-a)^3$ in the denominator leads to three terms, $\frac{A}{s-a} + \frac{B}{(s-a)^2} + \frac{C}{(s-a)^3}$ in the partial fractions expansion
- etc.,

Example: Using Partial Fraction Expansions

Suppose we know

$$\mathcal{L}[y] = \frac{2s+1}{s^2+3s+2}$$

We have

$$\begin{aligned}
\mathcal{L}[y] &= \frac{2s+1}{s^2-s+2} \\
&= \frac{2s+1}{(s+1)(s-2)} \\
&= \frac{A}{s+1} + \frac{B}{s-2} \\
&= A\frac{1}{s+1} + B\frac{1}{s-2} \\
&= A\mathcal{L}\left[e^{-x}\right] + B\mathcal{L}\left[e^{2x}\right] \\
&= \mathcal{L}\left[Ae^{-x} + Be^{2x}\right]
\end{aligned}$$

Now we just need to figure out the correct values for the constants A and B.

Let's go back to the Partial Fractions expansion:

$$\frac{2s+1}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

Multiplying both sides by $(s+1)(s-2)$ we have

$$2s+1 = A(s-2) + B(s+1)$$

This equation must be true for all values of s.
Choosing $s = -1$ yields

$$-2+1 = A(-3) + B(0) \quad \Longrightarrow \quad A = -\frac{1}{3}$$

Choosing $s = 2$ yields

$$4+1 = A(0) + B(3) \quad \Longrightarrow \quad B = \frac{5}{3}$$

Thus,

$$\frac{2s+1}{(s+1)(s-2)} = -\frac{1}{3}\frac{1}{s+1} + \frac{5}{3}\frac{1}{s-2}$$

And so

$$
\begin{aligned}
\mathcal{L}[y] &= \frac{2s+1}{s^2-s+2} \\
&= \frac{2s+1}{(s+1)(s-2)} \\
&= \frac{A}{s+1} + \frac{B}{s-2} \\
&= -\frac{1}{3}\frac{1}{s+1} + \frac{5}{3}\frac{1}{s-2} \\
&= -\frac{1}{3}\mathcal{L}\left[e^{-x}\right] + B\mathcal{L}\left[e^{2x}\right] \\
&= \mathcal{L}\left[-\frac{1}{3}e^{-x} + \frac{5}{3}e^{2x}\right]
\end{aligned}
$$

Thus,

$$y(x) = -\frac{1}{3}e^{-x} + \frac{5}{3}e^{2x}$$