Math 2233 - Lecture 14

Agenda:

1. Summary: Solving 2nd Order Linear ODEs
2. Examples
3. The Laplace Transform
4. The Laplace Transform Method for Solving ODEs

To solve

$$y'' + p(x)\, y' + q(x)\, y = g(x) \tag{1}$$

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5. The general solution of (1) is now

$$Y(x) = Y_p(x) + c_1 y_1(x) + c_2 y_2(x) \tag{5}$$

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$$\begin{aligned} y'' - 3y' + 2y &= 10 \\ y(0) &= 5 \\ y'(0) &= -1 \end{aligned}$$

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$$\begin{aligned} y_1(x) &= e^{2x} \\ y_2(x) &= e^{x} \end{aligned}$$

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and so

$$y_p(x) = -y_1(x)\int^x \frac{y_2(s)g(s)}{W\left[y_1, y_2\right](s)}ds + y_2(x)\int^x \frac{y_1(s)g(s)}{W\left[y_1, y_2\right](s)}ds$$

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Thus the general solution is

$$\begin{aligned} y(x) &= y_p(x) + c_1 y_1(x) + c_2 y_2(x) \\ &= 5 + c_1 e^{2x} + c_2 e^{x} \end{aligned}$$

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We have

$$\begin{aligned} 5 &= y(0) = 5 + c_1 + c_2 \quad \Rightarrow \quad c_1 + c_2 = 0 \\ -1 &= y'(0) = 2c_1 + c_2 \quad \Rightarrow \quad 2c_1 + c_2 = -1 \end{aligned}$$

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The solution to the initial value problem is thus

$$y(x) = 5 + e^{2x} - e^{x}$$

The Laplace Transform

Definition

The **Laplace transform** of a function $f(x)$ is

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We note that in the formula (6), s is the new variable upon which the Laplace transform $\mathcal{L}[f]$ depends.

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$$\begin{array}{cc} u = x^n & dv = e^{-sx} dx \\ \downarrow \frac{d}{dx} & \downarrow \int \\ du = n x^{n-1} dx & v = -\frac{1}{s} e^{-sx} \end{array}$$

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u = x^n & dv = e^{-sx}dx & \\
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du = nx^{n-1}dx & v = -\frac{1}{s}e^{-sx} &
\end{array}$$

$$\Rightarrow \qquad \mathcal{L}\left[x^n\right](s) = \left(x^n\right)\left(-\frac{1}{s}e^{-sx}\right)\Bigg|_{x=0}^{x=\infty} - \int_0^\infty nx^{n-1}\left(-\frac{1}{s}e^{-sx}\right)dx$$

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&= 0 - 0 + \frac{n}{s}\int x^{n-1}e^{-sx}dx \\
&= \frac{n}{s}\mathcal{L}\left[x^{n-1}\right]
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&= \frac{n}{s}\frac{n-1}{s}\cdots\frac{1}{s}\frac{1}{s} \\
&= \frac{n!}{s^{n+1}}
\end{aligned}$$

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(If $s \le b$ then the integral does not converge.)

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&= \tfrac{1}{a} + \tfrac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \\
&= \tfrac{1}{a} + \lim_{N\to\infty} \tfrac{s}{a} \left(-\tfrac{1}{a} e^{-sx} \sin(ax)\right)\Big|_0^N - \tfrac{s^2}{a^2} \int_0^\infty e^{-sx} \sin(ax)\, dx
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&= \tfrac{1}{a} + \tfrac{s}{a} \int_0^\infty e^{-sx} \cos(ax)\, dx \\
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&= \tfrac{1}{a} + 0 - \tfrac{s^2}{a^2} L[f](s) \quad ,
\end{aligned}$$

Example 4

If

$$f(x) = \sin(ax)$$

then, integrating by twice by parts,

$$\begin{array}{rcl}
\mathcal{L}[f](s) & = & \int_0^\infty \sin(ax) e^{-sx}\, dx \\
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we find

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(If $s \leq 0$, the integral on the first line does not converge, so $\mathcal{L}[f](s)$ is only defined for $s > 0$.)

A Table of Basic Laplace Transforms

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\mathcal{L}\left[t^n e^{at}\right] = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2+b^2}$$

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(i) $\mathcal{L}[c_1 f_1 + c_2 f_2] = c_1 \mathcal{L}[f_1] + c_2 \mathcal{L}[f_2]$

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Application of Laplace Transforms to Initial Value Problems

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Thus

$$\left(s^2\mathcal{L}[y] - 2s - 4\right) - (s\mathcal{L}[y] - 2) - 2\mathcal{L}[y] = 0$$

or

$$\left(s^2 - s - 2y\right) \mathcal{L}\left[y\right] - 2s - 2 = 0$$

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We conclude

$$y\left(x\right) = 2e^{2x}$$

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In what follows, we shall be concentrating on Step 4.

Inverting the Laplace Transforms of Rational Functions

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or

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Digression: Review of Partial Fractions

Partial fraction expansions are a very useful tool for figuring out inverse Laplace transforms. A simple way to understand Partial Fractions expansions is that they reverse the algebra that goes into putting a sum of rational functions over a common denominator. For example,

$$\frac{2}{s+1} + \frac{3}{s-2} = \frac{2\,(s-2) + 3\,(s+1)}{(s+1)\,(s-2)} = \frac{5s-1}{(s+1)\,(s-2)}$$

or

$$\frac{5s-1}{(s+1)\,(s-2)} = \frac{2}{s+1} + \frac{3}{s-2}$$

In the latter equation, the right hand side is the Partial Fractions Expansion of $\frac{5s-1}{(s+1)(s-2)}$.

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and $P(s)$ *is a polynomial such that* $\deg(P) < \deg(Q)$ *.*
Then there exists numbers $a_{ij}, i = 1..k, j = 1, \ldots, m_i$*, such that*

$$\frac{P(s)}{Q(s)} = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{a_{ij}}{(s - a_i)^j}$$

Here is an example as to how this theorem is applied.

$$\frac{s^2 - 3s + 1}{(s-3)(s+2)^3} = \frac{a_{11}}{s-3} + \frac{a_{21}}{s+2} + \frac{a_{22}}{(s+2)^2} + \frac{a_{23}}{(s+2)^3}$$

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(I'll discuss below how to find the correct values for these numbers.)
Rather than introducing indexed symbols a_{ij}. One typically just uses different letters to represent the numbers in the numerators on the right; e.g., as in

$$\frac{s^2-3s+1}{(s-3)(s+2)^3}=\frac{A}{s-3}+\frac{B}{s+2}+\frac{C}{(s+2)^2}+\frac{D}{(s+2)^3} \qquad (*)$$

Also, rather than relying on the formula in the theorem, one usually constructs the Partial Fractions Expansion by simply adding up the contributions for each factor $\frac{1}{(s-a)^m}$ in $F(s)$ in the denominator $Q(x)$;

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- a factor $(s-a)$ in the denominator leads to a term of the form $\frac{A}{s-z}$ in the partial fractions expansion
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Now we just need to figure out the correct values for the constants A and B.

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$$\frac{2s+1}{(s+1)(s-2)} = -\frac{1}{3}\frac{1}{s+1} + \frac{5}{3}\frac{1}{s-2}$$

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Thus,

$$y(x) = -\frac{1}{3}e^{-x} + \frac{5}{3}e^{2x}$$