Math 2233 - Lecture 15

Agenda

1. Scheduling 2nd Exam
2. Laplace Transform Method
3. Inverting Laplace Transforms
   - Case (i) Denominator Factorizes
   - Case (ii) Denominator is a Sum of Squares
   - Case (iii) Denomminator is a Difference of Squares
4. Examples

The Laplace Transform

Definition

The **Laplace transform** of a function $f(x)$ is

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x)\,dx \quad . \tag{1}$$

Theorem

(i) $\mathcal{L}\left[c_1 f_1 + c_2 f_2\right] = c_1 \mathcal{L}\left[f_1\right] + c_2 \mathcal{L}\left[f_2\right]$

(ii) $\mathcal{L}\left[\frac{df}{dx}\right] = s\mathcal{L}\left[f\right] - f\left(0\right)$

(iii) $\mathcal{L}\left[\frac{d^2 f}{dx^2}\right] = s^2 \mathcal{L}\left[f\right] - sf\left(0\right) - f'\left(0\right)$

A Table of Basic Laplace Transforms

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\mathcal{L}\left[t^n e^{at}\right] = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\left[e^{at}\cos(bt)\right] = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\left[e^{at}\sinh(bt)\right] = \frac{b}{(s-a)^2-b^2}$$

$$\mathcal{L}\left[e^{at}\cosh(bt)\right] = \frac{s-a}{(s-a)^2-b^2}$$

Example: Using the Laplace Transform to Solve an Initial Value Problem

Consider the following initial value problem

$$\begin{aligned} y'' + 5y' + 6y &= 0 \\ y(0) &= 0 \\ y'(0) &= 2 \end{aligned}$$

We begin by taking the Laplace transform of the differential equation term-by-term

$$\begin{aligned} 0 &= \mathcal{L}\left[y''\right] + 5\mathcal{L}\left[y'\right] + 6\mathcal{L}\left[y\right] \\ &= s^2\mathcal{L}\left[y\right] - sy(0) - y'(0) \\ &\quad +5\left(s\mathcal{L}\left[y\right] - y(0)\right) \\ &\quad +6\mathcal{L}\left[y\right] \\ &= \left(s^2 + 5s + 6\right)\mathcal{L}\left[y\right] - s(0) - 2 - (5)(0) \\ &= (s+2)(s+3)\mathcal{L}\left[y\right] - 2 \end{aligned}$$

Solving for $\mathcal{L}[y]$, we get

$$\mathcal{L}[y] = \frac{2}{(s+2)(s+3)} \tag{3}$$

We now know the Laplace transform of the solution to the I.V.P.; but we do not yet know exactly what the solution is.
So we now have to figure what function has

$$\frac{2}{(s+2)(s+3)}$$

as its Laplace transform

We'll now manipulate the right hand side of (3) until we can recognize it as a Laplace transform of a particular function.

Inverting the Laplace Transform using Partial Fraction Expansions

As an ansatz for a Partial Fractions Expansion of the right hand side of (3) we have

$$\frac{2}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} \tag{4}$$

with the constants A and B to be determined.
To determine A and B we multiply both sides of (4) by $(s+2)(s+3)$ we have

$$2 = (s+3)\,A + (s+2)\,B \tag{5}$$

This equation must be true for all values of s.
Substituting the particular value $s = -3$ into (5) yields

$$2 = (0)A + (-1)\,B \qquad \Rightarrow \quad B = -2$$

Substituting the particular value $s = -2$ into (5) yields

$$2 = (1)\,A + (0)B \qquad \Rightarrow \quad A = 2$$

Thus,

$$\mathcal{L}[y] = \frac{2}{(s+2)(s+3)} = 2\frac{1}{s+2} - 2\frac{1}{s+3}$$

Consulting our Table of Laplace Transforms, we see

$$\mathcal{L}[e^{ax}] = \frac{1}{s-a}$$

and so

$$\begin{aligned} \mathcal{L}[y] &= 2\mathcal{L}\left[e^{-2x}\right] - 2\mathcal{L}\left[e^{-3x}\right] \\ &= \mathcal{L}\left[2e^{-2x} - 2e^{-3x}\right] \end{aligned}$$

and so our solution is

$$y(x) = 2e^{-2x} - 2e^{-3x}$$

Hints for Inverting Laplace Transforms of $F(s) = \frac{P(s)}{Q(s)}$

Suppose you need to find the function $y(x)$ such that

$$\mathcal{L}[y] = F(s) = \frac{P(s)}{Q(s)}$$

with $P(s)$ and $Q(s)$ polynomials.

Case 1: $Q(s)$ factorizes completely

In this case, apply a Partial Fractions Expansion to re-express the $\mathcal{L}[y]$ as a linear combination of terms of the form $\frac{1}{(s-a)^k}$

Each of these terms should be recognizable as a constant times one of the following Laplace transforms

$$\begin{aligned}
\frac{1}{s^{n+1}} &= \frac{1}{n!}\mathcal{L}[t^n] \\
\frac{1}{s-a} &= \mathcal{L}\left[e^{at}\right] \\
\frac{1}{(s-a)^{n+1}} &= \frac{1}{n!}\mathcal{L}\left[t^n e^{at}\right]
\end{aligned}$$

and the whole thing can be recombined to express $\mathcal{L}[y]$ into a single Laplace transform.

Example: Inverting a Laplace transform via Partial Fractions Expansions, Cont'd

In summary:

If $Q(s)$ can be easily factorized in terms of linear factors of the form $(s-a)$

- Replace $\mathcal{L}[y](s)$ by its partial fractions expansion;
- Identify each term in the P.F.E. as a constant times a known Laplace transform and then
- Use the linearity of the Laplace transform to consolidate the the result into a single Laplace transform.
- Identify the function inside the single Laplace transform as $y(x)$

Example: Inverting a Laplace transform via Partial Fractions Expansions

Determine the function $y(x)$ whose Laplace transform is

$$\mathcal{L}[y](s) = \frac{s^2+1}{(s+1)^2(s+2)}$$

As an ansatz for the Partial Fractions Expansion of the right hand side, we have

$$\frac{s^2+1}{(s+1)^2(s+2)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s+2}$$

Multiplying both sides by $(s+1)^2(s+2)$ yields

$$s^2+1 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \qquad (*)$$

Imposing $s=-1$ on $(*)$ yields

$$1+1 = A(0) + B(1) + C(0) \quad \Rightarrow \quad B = 2$$

Example: Inverting a Laplace transform via Partial Fractions Expansions, Cont'd

Imposing $s = -2$ on the same equation (*) yields

$$4 + 1 = A(0) + B(0) + C(1) \quad \Rightarrow \quad C = 5$$

We need one more equation to determine the unknown coefficient A. We'll try $s = 0$.

$$0 + 1 = 2A + 2B + C$$

or

$$1 = 2A + 4 + 5 \quad \Rightarrow \quad A = -4$$

Thus

$$\begin{aligned}
\mathcal{L}[y](s) &= \frac{s^2+1}{(s+1)^2(s+2)} \\
&= -4\frac{1}{s+1} + 2\frac{1}{(s+1)^2} + 5\frac{1}{s+2} \\
&= -4\mathcal{L}\left[e^{-x}\right] + 2\left(\frac{1}{1!}\mathcal{L}\left[t^1 e^{-x}\right]\right) + 5\mathcal{L}\left[e^{-2x}\right]
\end{aligned}$$

or

$$\mathcal{L}[y] = \mathcal{L}\left[-4e^{-x} + 2xe^{-x} + 5e^{-2x}\right]$$

and so

$$y(x) = -4e^{-x} + 2xe^{-x} + 5e^{-2x}$$

Hints for Inverting Laplace Transforms of $F(s) = \frac{P(s)}{Q(s)}$, Cont'd

Case 2: $Q(s)$ is a sum of squares
Suppose

$$\mathcal{L}[y] = \frac{P(s)}{Q(s)}$$

with

$$Q(s) = (s-a)^2 + b^2$$

and degree $P(s) < 2$.

For this situation, we'll try to view $\mathcal{L}[y]$ as a linear combination of the following two Laplace transforms.

$$\begin{aligned} \mathcal{L}\left[e^{at}\sin(bt)\right] &= \frac{b}{(s-a)^2 + b^2} \\ \mathcal{L}\left[e^{at}\cos(bt)\right] &= \frac{s-a}{(s-a)^2 + b^2} \end{aligned}$$

Hints for Inverting Laplace Transforms of $F(s) = \frac{P(s)}{Q(s)}$, Cont'd

And so we set

$$F(s) = A\frac{b}{(s-a)^2 + b^2} + B\frac{s-a}{(s-a)^2 + b^2}$$

and try to figure out the appropriate values for A and B.
Once we find the correct values for A and B, we'll then have

$$\begin{aligned}
\mathcal{L}[y] &= A\mathcal{L}\left[e^{ax}\sin(bx)\right] + B\mathcal{L}\left[e^{ax}\cos(bx)\right] \\
&= \mathcal{L}\left[Ae^{ax}\sin(bx) + Be^{ax}\cos(bx)\right]
\end{aligned}$$

and so

$$y = Ae^{ax}\sin(bx) + Be^{ax}\cos(bx)$$

Hints for Inverting Laplace Transform of $F(s) = \frac{P(s)}{Q(s)}$, Cont'd

Case 3: $Q(s)$ is a difference of squares
Suppose

$$Q(s) = (s-a)^2 - b^2$$

and degree $P(s) < 2$.

For this situation, we'll try to view $\mathcal{L}[y]$ as a linear combination of the following two Laplace transforms.

$$\begin{aligned} \mathcal{L}\left[e^{at}\sinh(bt)\right] &= \frac{b}{(s-a)^2 - b^2} \\ \mathcal{L}\left[e^{at}\cosh(bt)\right] &= \frac{s-a}{(s-a)^2 - b^2} \end{aligned}$$

Hints for Inverting Laplace Transforms of $F(s) = \frac{P(s)}{Q(s)}$, Cont'd

And so we set

$$F(s) = A\frac{b}{(s-a)^2 - b^2} + B\frac{s-a}{(s-a)^2 - b^2}$$

and try to figure out the appropriate values for A and B.
Once we find the correct values for A and B, we'll then have

$$\begin{aligned}\mathcal{L}[y] &= A\mathcal{L}[e^{ax}\sinh(bx)] + B\mathcal{L}[e^{ax}\cosh(bx)] \\ &= \mathcal{L}[Ae^{ax}\sinh(bx) + Be^{ax}\cosh(bx)]\end{aligned}$$

and so

$$y = Ae^{ax}\sinh(bx) + Be^{ax}\cosh(bx)$$

Example 1

$$\begin{aligned} y'' + 4y + 5 &= 0 \\ y(0) &= 1 \\ y'(0) &= 2 \end{aligned}$$

We have

$$\begin{aligned} \left(s^2\mathcal{L}[y] - sy(0) + y'(0)\right) + 4\left(s\mathcal{L}[y] - y(0)\right) + 5\mathcal{L}[y] &= 0 \\ \left(s^2 + 4s + 5\right)\mathcal{L}[y] + s - 2 - 4 &= 0 \end{aligned}$$

or

$$\mathcal{L}[y] = \frac{6 - s}{s^2 + 4s + 5}$$

The denominator does not factorize easily, and so we'll try to re-express it as a sum or difference of squares.

Example 1, Cont'd

Noting that

$$s^2+4s+5 = s^2+4s+(4-4)+5 = \left(s^2+4s+4\right)+1 = (s+2)^2+(1)^2$$

we see that the denominator is a sum of squares, so we'll try to reconstruct $\mathcal{L}\left[y\right]$ from

$$\begin{aligned} \mathcal{L}\left[e^{ax}\sin\left(bx\right)\right] &= \frac{b}{(s-a)^2+b^2} \\ \mathcal{L}\left[e^{ax}\cos\left(bx\right)\right] &= \frac{s-a}{(s-a)^2+b^2} \end{aligned}$$

with $a=-2$ and $b=1$We thus set

$$\frac{6-s}{s^2+4s+5} = A\frac{1}{(s+2)^2+1} + B\frac{s+2}{(s+2)^2+1}$$

Multiplying both sides by $s^2+4s+1 = \left(s+2^2\right)+1$, we get

$$6-s = A + B\left(s+2\right)$$

Example 1, Cont'd

$$6 - s = A + B\left(s+2\right)$$

Now $s = -2$ implies

$$A = 8$$

and $s = 0$ implies

$$6 = A + 2B \quad \Rightarrow \quad 2B = 6 - A = -2 \quad \Rightarrow \quad B = -1$$

Thus,

$$\begin{aligned}
\mathcal{L}\left[y\right] &= \frac{6-s}{s^2+4s+5} = A\frac{1}{\left(s+2\right)^2+1} + B\frac{s+2}{\left(s+2\right)^2+1} \\
&= 8\frac{1}{\left(s+2\right)^2+1} - \frac{s+2}{\left(s+2\right)^2+1} \\
&= 8\mathcal{L}\left[e^{-2x}\sin\left(x\right)\right] - \mathcal{L}\left[e^{-2x}\cos\left(x\right)\right] \\
&= \mathcal{L}\left[8e^{-2x}\sin\left(x\right) - e^{-2x}\cos\left(x\right)\right]
\end{aligned}$$

and so

$$y\left(x\right) = 8e^{-2x}\sin\left(x\right) - e^{-2x}\cos\left(x\right)$$

Example 2

$$\begin{aligned} y'' + 3y' + 2y &= x \\ y(0) &= 0 \\ y'(0) &= 4 \end{aligned}$$

Note that this is a nonhomogeneous ODE. The Laplace transform method still works for this case.

$$\begin{aligned} \mathcal{L}\left[y''\right] + 3\mathcal{L}\left[y'\right] + 2\mathcal{L}[y] &= \mathcal{L}[x] \\ \left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 3\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] &= \frac{1}{s^2} \\ \left(s^2 + 3s + 2\right)\mathcal{L}[y] - 4 &= \frac{1}{s^2} \end{aligned}$$

so

$$\mathcal{L}[y] = \frac{4}{s^2\left(s^2 + 3s - 4\right)} = \frac{4}{s^2(s-1)(s+4)}$$

Seeing that the denominator of $\mathcal{L}[y]$ can be completely factored, we can now carry out a partial fractions expansion of the right hand side

Example 2, Cont'd

$$\mathcal{L}[y] = \frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

$$\Rightarrow \quad 4 = As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$$

or

$$\begin{aligned} s &= 0 \quad \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2 \\ s &= -1 \quad \Rightarrow \quad 4 = 3C \quad \Rightarrow \quad C = \frac{4}{3} \\ s &= -2 \quad \Rightarrow \quad 4 = -4D \quad \Rightarrow \quad D = -1 \end{aligned}$$

We need one more equation to determine A. We can use $s = 1$.

$$\begin{aligned} 4 &= 6A + 6B + 3C + 2D = 6A + 6(2) + 3\left(\frac{4}{3}\right) + 2(-1) \\ &\Rightarrow \quad 4 = 6A + 14 \\ &\Rightarrow \quad A = -\frac{5}{3} \end{aligned}$$

Example 2, Cont'd

Thus,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \\
&= -\frac{5}{3}\frac{1}{s} + 2\frac{1}{s^2} + \frac{4}{3}\frac{1}{s+1} - \frac{1}{s+2} \\
&= -\frac{5}{3}\mathcal{L}[1] + 2\mathcal{L}[x] + \frac{4}{3}\mathcal{L}\left[e^{-x}\right] - \mathcal{L}\left[e^{-2x}\right] \\
&= \mathcal{L}\left[-\frac{5}{3} + 2x + \frac{4}{3}e^{-x} - e^{-2x}\right]
\end{aligned}$$

and so

$$y(x) = -\frac{5}{3} + 2x + \frac{4}{3}e^{-x} - e^{-2x}$$