Math 2233 - Lecture 15

Agenda

1. Scheduling 2nd Exam
2. Laplace Transform Method
3. Inverting Laplace Transforms
   - Case (i) Denominator Factorizes
   - Case (ii) Denominator is a Sum of Squares
   - Case (iii) Denomminator is a Difference of Squares
4. Examples

The Laplace Transform

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Definition

The **Laplace transform** of a function $f(x)$ is

$$\mathcal{L}[f](s) = \int_0^\infty e^{-sx} f(x)\, dx \quad . \tag{1}$$

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(ii) $\mathcal{L}\left[\frac{df}{dx}\right] = s\mathcal{L}\left[f\right] - f\left(0\right)$

(iii) $\mathcal{L}\left[\frac{d^2 f}{dx^2}\right] = s^2\mathcal{L}\left[f\right] - sf\left(0\right) - f'\left(0\right)$

A Table of Basic Laplace Transforms

$$\mathcal{L}\left[t^n\right] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

$$\mathcal{L}\left[t^n e^{at}\right] = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2+b^2}$$

$$\mathcal{L}\left[e^{at}\cos(bt)\right] = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathcal{L}\left[e^{at}\sinh(bt)\right] = \frac{b}{(s-a)^2-b^2}$$

$$\mathcal{L}\left[e^{at}\cosh(bt)\right] = \frac{s-a}{(s-a)^2-b^2}$$

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$$\begin{aligned} 0 &= \mathcal{L}[y''] + 5\mathcal{L}[y'] + 6\mathcal{L}[y] \\ &= s^2\mathcal{L}[y] - sy(0) - y'(0) \\ &\quad +5(s\mathcal{L}[y] - y(0)) \\ &\quad +6\mathcal{L}[y] \end{aligned}$$

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Solving for $\mathcal{L}[y]$, we get

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We'll now manipulate the right hand side of (3) until we can recognize it as a Laplace transform of a particular function.

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and so our solution is

$$y(x) = 2e^{-2x} - 2e^{-3x}$$

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with $P(s)$ and $Q(s)$ polynomials.

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\end{aligned}$$

and the whole thing can be recombined to express $\mathcal{L}[y]$ into a single Laplace transform.

Example: Inverting a Laplace transform via Partial Fractions Expansions, Cont'd

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If $Q(s)$ can be easily factorized in terms of linear factors of the form $(s-a)$

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- Use the linearity of the Laplace transform to consolidate the the result into a single Laplace transform.
- Identify the function inside the single Laplace transform as $y(x)$

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$$s^2+1 = A(s+1)(s+2) + B(s+2) + C(s+1)^2 \qquad (*)$$

Imposing $s=-1$ on (*) yields

$$1+1 = A(0) + B(1) + C(0) \quad \Rightarrow \quad B = 2$$

Example: Inverting a Laplace transform via Partial Fractions Expansions, Cont'd

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Imposing $s = -2$ on the same equation (*) yields

$$4 + 1 = A(0) + B(0) + C(1) \quad \Rightarrow \quad C = 5$$

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Thus

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$$\begin{aligned} \mathcal{L}[y](s) &= \frac{s^2+1}{(s+1)^2(s+2)} \\ &= -4\frac{1}{s+1} + 2\frac{1}{(s+1)^2} + 5\frac{1}{s+2} \end{aligned}$$

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Thus

$$\begin{aligned}
\mathcal{L}[y](s) &= \frac{s^2 + 1}{(s+1)^2 (s+2)} \\
&= -4\frac{1}{s+1} + 2\frac{1}{(s+1)^2} + 5\frac{1}{s+2} \\
&= -4\mathcal{L}\left[e^{-x}\right] + 2\left(\frac{1}{1!}\mathcal{L}\left[t^1 e^{-x}\right]\right) + 5\mathcal{L}\left[e^{-2x}\right]
\end{aligned}$$

or

$$\mathcal{L}[y] = \mathcal{L}\left[-4e^{-x} + 2xe^{-x} + 5e^{-2x}\right]$$

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and so

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Suppose

$$\mathcal{L}[y] = \frac{P(s)}{Q(s)}$$

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and degree $P(s) < 2$.

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For this situation, we'll try to view $\mathcal{L}[y]$ as a linear combination of the following two Laplace transforms.

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with

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and degree $P(s) < 2$.

For this situation, we'll try to view $\mathcal{L}[y]$ as a linear combination of the following two Laplace transforms.

$$\mathcal{L}\left[e^{at}\sin(bt)\right] = \frac{b}{(s-a)^2 + b^2}$$

Hints for Inverting Laplace Transforms of $F(s) = \frac{P(s)}{Q(s)}$, Cont'd

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Hints for Inverting Laplace Transforms of $F(s) = \frac{P(s)}{Q(s)}$, Cont'd

And so we set

$$F(s) = A\frac{b}{(s-a)^2 + b^2} + B\frac{s-a}{(s-a)^2 + b^2}$$

and try to figure out the appropriate values for A and B.

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Example 1

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The denominator does not factorize easily, and so we'll try to re-express it as a sum or difference of squares.

Example 1, Cont'd

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Noting that

$$s^2+4s+5 = s^2+4s+(4-4)+5 = \left(s^2+4s+4\right)+1 = (s+2)^2+(1)^2$$

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we see that the denominator is a sum of squares, so we'll try to reconstruct $\mathcal{L}[y]$ from

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with $a=-2$ and $b=1$.

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$$\frac{6-s}{s^2+4s+5} = A\frac{1}{(s+2)^2+1} + B\frac{s+2}{(s+2)^2+1}$$

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Multiplying both sides by $s^2+4s+1 = (s+2)^2+1$, we get

$$6-s = A + B(s+2)$$

Example 1, Cont'd

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\mathcal{L}[y] &= \frac{6-s}{s^2+4s+5} = A\frac{1}{(s+2)^2+1} + B\frac{s+2}{(s+2)^2+1} \\
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and so

$$y(x) = 8e^{-2x}\sin(x) - e^{-2x}\cos(x)$$

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\mathcal{L}\left[y''\right] + 3\mathcal{L}\left[y'\right] + 2\mathcal{L}\left[y\right] = \mathcal{L}\left[x\right]
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$$\begin{aligned} \mathcal{L}\left[y''\right] + 3\mathcal{L}\left[y'\right] + 2\mathcal{L}[y] &= \mathcal{L}[x] \\ \left(s^2\mathcal{L}[y] - sy(0) - y'(0)\right) + 3\left(s\mathcal{L}[y] - y(0)\right) + 2\mathcal{L}[y] &= \frac{1}{s^2} \end{aligned}$$

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so

$$\mathcal{L}[y] = \frac{4}{s^2\left(s^2 + 3s - 4\right)} = \frac{4}{s^2\left(s - 1\right)\left(s + 4\right)}$$

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so

$$\mathcal{L}[y] = \frac{4}{s^2\left(s^2 + 3s - 4\right)} = \frac{4}{s^2(s-1)(s+4)}$$

Seeing that the denominator of $\mathcal{L}[y]$ can be completely factored, we can now carry out a partial fractions expansion of the right hand side

Example 2, Cont'd

$$\mathcal{L}\left[y\right] = \frac{4}{s^2\left(s+1\right)\left(s+2\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

Example 2, Cont'd

$$\mathcal{L}[y] = \frac{4}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

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$$\begin{aligned} s &= 0 \quad \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2 \\ s &= -1 \quad \Rightarrow \quad 4 = 3C \quad \Rightarrow \quad C = \frac{4}{3} \end{aligned}$$

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We need one more equation to determine A.

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We need one more equation to determine A. We can use $s = 1$.

$$4 = 6A + 6B + 3C + 2D = 6A + 6(2) + 3\left(\frac{4}{3}\right) + 2(-1)$$

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$$\begin{aligned} 4 &= 6A + 6B + 3C + 2D = 6A + 6(2) + 3\left(\frac{4}{3}\right) + 2(-1) \\ \Rightarrow & \quad 4 = 6A + 14 \end{aligned}$$

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$$\Rightarrow \quad 4 = As(s+1)(s+2) + B(s+1)(s+2) + Cs^2(s+2) + Ds^2(s+1)$$

or

$$\begin{aligned} s &= 0 \quad \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2 \\ s &= -1 \quad \Rightarrow \quad 4 = 3C \quad \Rightarrow \quad C = \frac{4}{3} \\ s &= -2 \quad \Rightarrow \quad 4 = -4D \quad \Rightarrow \quad D = -1 \end{aligned}$$

We need one more equation to determine A. We can use $s = 1$.

$$\begin{aligned} 4 &= 6A + 6B + 3C + 2D = 6A + 6(2) + 3\left(\frac{4}{3}\right) + 2(-1) \\ &\Rightarrow \quad 4 = 6A + 14 \\ &\Rightarrow \quad A = -\frac{5}{3} \end{aligned}$$

Example 2, Cont'd

Example 2, Cont'd

Thus,

$$\mathcal{L}[y] = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

Example 2, Cont'd

Thus,

$$\begin{aligned}\mathcal{L}[y] &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \\ &= -\frac{5}{3}\frac{1}{s} + 2\frac{1}{s^2} + \frac{4}{3}\frac{1}{s+1} - \frac{1}{s+2}\end{aligned}$$

Example 2, Cont'd

Thus,

$$
\begin{aligned}
\mathcal{L}[y] &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \\
&= -\frac{5}{3}\frac{1}{s} + 2\frac{1}{s^2} + \frac{4}{3}\frac{1}{s+1} - \frac{1}{s+2} \\
&= -\frac{5}{3}\mathcal{L}[1] + 2\mathcal{L}[x] + \frac{4}{3}\mathcal{L}\left[e^{-x}\right] - \mathcal{L}\left[e^{-2x}\right]
\end{aligned}
$$

Example 2, Cont'd

Thus,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \\
&= -\frac{5}{3}\frac{1}{s} + 2\frac{1}{s^2} + \frac{4}{3}\frac{1}{s+1} - \frac{1}{s+2} \\
&= -\frac{5}{3}\mathcal{L}[1] + 2\mathcal{L}[x] + \frac{4}{3}\mathcal{L}\left[e^{-x}\right] - \mathcal{L}\left[e^{-2x}\right] \\
&= \mathcal{L}\left[-\frac{5}{3} + 2x + \frac{4}{3}e^{-x} - e^{-2x}\right]
\end{aligned}$$

and so

Example 2, Cont'd

Thus,

$$\begin{aligned}
\mathcal{L}[y] &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2} \\
&= -\frac{5}{3}\frac{1}{s} + 2\frac{1}{s^2} + \frac{4}{3}\frac{1}{s+1} - \frac{1}{s+2} \\
&= -\frac{5}{3}\mathcal{L}[1] + 2\mathcal{L}[x] + \frac{4}{3}\mathcal{L}\left[e^{-x}\right] - \mathcal{L}\left[e^{-2x}\right] \\
&= \mathcal{L}\left[-\frac{5}{3} + 2x + \frac{4}{3}e^{-x} - e^{-2x}\right]
\end{aligned}$$

and so

$$y(x) = -\frac{5}{3} + 2x + \frac{4}{3}e^{-x} - e^{-2x}$$