Math 2233 - Lecture 16

Agenda

1. 2nd Order Linear ODEs (general case)
2. Taylor Series Method
3. Power Series Method

Review: Solving $y'' + p(x)y' + q(x)y = g(x)$

1. Find at least 1 independent solution $y_1(x)$ of

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{0}$$

2. If only 1 solution is found in Step 1, calculate a 2nd independent solution of (0)

$$y_2(x) = y_1(x) \int \frac{1}{(y_1(x))^2} \exp\left[-\int p(x)\,dx\right] dx \tag{2}$$

3. The general solution of (0) is now

$$y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{3}$$

4. Calculate 1st solution Y_p of (1)

$$Y_p(x) = -y_1(x) \int \frac{y_2(x)\,g(x)}{W[y_1, y_2](x)} dx + y_2(x) \int \frac{y_1(x)\,g(x)}{W[y_1, y_2](x)} dx \tag{4}$$

5. The general solution of (1) is now

$$Y(x) = Y_p(x) + c_1 y_1(x) + c_2 y_2(x) \tag{5}$$

Upshot:

The crux of the problem of solving

$$y'' + p(x)y' + q(x)y = g(x) \tag{1}$$

boils down to finding **one** solution of the corresponding homogenoeus linear ODE

$$y'' + p(x)y' + q(x)y = 0 \tag{0}$$

But, at this point, we only know how to solve two special cases of (0)

- Constant Coefficients Case : $ay'' + by' + cy = 0$
- Euler-type Case: $ax^2y'' + bxy' + cy = 0$

The principal goal for the rest of the course will be to develop a means of attacking the general case of Eq. (0) head on.

Making an Ansatz for the First Solution

We were able to find solutions to the Constant Coefficient and Euler-type ODEs because we were clever enough to guess some reasonable trial solutions that actually worked.

$$
\begin{aligned}
ay'' + by' + cy &= 0 \quad \Rightarrow \quad \text{try } y(x) = e^{\lambda x} \\
ax^2y'' + bxy' + cy &= 0 \quad \Rightarrow \quad \text{try } y(x) = x^r
\end{aligned}
$$

However, the equation

$$
y'' + p(x)y' + q(x)y = 0 \tag{0}
$$

is too general to allow us to simply guess what the solution should look like.

Or so one might think.

But there is a particular way of writing a smooth function $f(x)$ which is, in a certain sense, has a similar form no matter what $f(x)$ is.

(Remark: a function $f(x)$ is **smooth** at a point x if f is continuous at x and all of the derivatives $\dfrac{d^nf}{dx^n}$ are defined at the point x.)

Taylor Series

Theorem (Taylor Expansion Theorem)

Let $f(x)$ be a function that is smooth within a neighborhood of the point x_0. Then

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n \\ &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots \end{aligned}$$

at least on a neighborhood of x_0.

In Calculus II, Taylor's Theorem is motivated mostly by goal of finding a easy way to approximate an arbitrary function of x by a finite polynomial.

However, here in Math 2233, our goal will be to interpret the infinite series on right hand side as a simply another way of writing down a formula for the function $f(x)$.

In this interpretation, once we know the numbers

$$\frac{f^{(n)}(x_0)}{n!} \quad ,$$

the equality

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

tells us **exactly** what the function $f(x)$ is near x_0 – so long as we account for all of the terms on the right hand side.

What I show you next is that if $f(x)$ obeys an ODE of the form

$$y'' + p(x)y' + q(x)y = 0 \qquad (0)$$

then it is possible to determine all the numbers $\frac{f^{(n)}(x_0)}{n!}$ explicitly. In fact, we will solve differential equations of the form (0) by determining the Taylor expansions of solutions.

Example: Taylor Series Method

Find the first five terms of the Taylor expansion about $x = 0$ of the solution to

$$\begin{aligned} y'' + 2xy' + y &= 0 \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

The Taylor expansion of the solution $y(x)$ about $x = 0$ is given by the formula

$$y(x) = y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \cdots$$

To make this explicit, we need to figure out numerical values for $y(0), y'(0), y''(0), \ldots$. Now the values of $y(0)$ and $y'(0)$ are determined by the initial conditions

$$\begin{aligned} y(0) &= 1 \\ y'(0) &= 0 \end{aligned}$$

Example, Cont'd

To determine $y''(0)$, we can evaluate the differential equation itself at $x = 0$:

$$\begin{aligned} y''(0) + 2xy'(0) + y(0) &= 0 \\ \implies \quad y''(0) &= \left(-2xy' - y\right)|_{x=0}) = 0 - y(0) \\ &= -1 \end{aligned}$$

To get a value for $y'''(0)$, we can differentiate the differential equation and evaluate the result at $x = 0$:

$$\begin{aligned} y'''(0) &= \left(-2y'(x) - 2xy''(x) - y'(x)\right)|_{x=0} \\ &= 0 - 0 - 0 \;=\; 0 \end{aligned}$$

To get a value for $y^{iv}(0)$ we differentiate the differential equation again:

$$\begin{aligned} y^{iv}(0) &= \left(-2y''(x) - 2y''(x) - 2xy'''(x) - y''(x)\right)|_{x=0} \\ &= -2(-1) - 2(-1) - 0 - (-1) \;=\; 5 \end{aligned}$$

Example, Cont'd

Thus, to order x^4

$$\begin{aligned} y(x) &= y(0) + y'(0)x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{iv}(0)}{4!}x^4 + \cdots \\ &= 1 + 0x - \frac{1}{2}x^2 - \frac{0}{6}x^3 + \frac{5}{24}x^4 + \cdots \\ &= 1 - \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots \end{aligned}$$

$\square$

Let's now condense our notation a bit by setting

$$a_n = \frac{f^{(n)}(x_o)}{n!} \tag{6}$$

so that a Taylor expansion can be expressed as

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \tag{7}$$

If we had an explicit formula for $f(x)$ then obviously we could compute each of the coefficients a_n in its Taylor expansion using equation (6). On the other hand, if we have formulas for all the coefficients a_n then can still write down the Taylor expansion of $f(x)$ via (3) and so we have effectively determined $f(x)$. The point of all this is that every smooth function can be expressed in the form (7) and by determining all the values of the constants a_n you effectively specify $f(x)$.

Power Series Nomenclature

When

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_o)^n \tag{8}$$

- the infinite summation on the right is called the **power series** expression for $y(x)$
- the constants a_n are called the **coefficients** of the power series
- the number x_0 is referred to as the **expansion point**
- the expressions $a_n \left(x - x_0\right)^n$ are the individual **terms** of the power series

Now I can state our strategy for solving a general second order linear differential equation

$$y'' + p(x)y' + q(x)y = 0$$

We shall assume that our solution is a smooth function and so it has a Taylor expansion about a point x_0:

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \tag{8}$$

We'll then plug this power series expression for $y(x)$ into the differential equation and try to determine the coefficients a_n by demanding that (8) actually solves the ODE. So the parameters $a_0, a_1, a_2, \cdots$ will be used like the parameter λ was used in constant coefficients case (where we looked for solutions of the form $y(x) = e^{\lambda x}$), or like the parameter r in the Euler-typecase (where we looked for solutions of the form $y(x) = x^r$).

Some Basic Facts about Convergent Power Series

An expression of the form

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n \qquad (7)$$

is called a **formal power series**. The reason for the qualification "formal" is because the summation over infinitely many terms can not actually be carried out (contrary to what the notation suggests) However, if it happens that

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n \left(x - x_0\right)^n \quad \text{exists for all } x \text{ close enough to } x_0$$

then we say that (7) is a **convergent power series**, and the expression (7) is interpretable as a legitimate function of x:

$$f(x) \equiv \lim_{N\to\infty} \sum_{n=0}^{N} a_n \left(x - x_0\right)^n$$

Some Basic Facts about Convergent Power Series, Cont'd

Theorem

Let $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ be a convergent power series.

(i) *If $f(x) = 0$ for all x, then $a_n = 0$ for all n.*

(ii) *$f(x)$ is differentiable and*

$$\frac{df}{dx}(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

is another convergent power series.

(iii) *If $g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$ is another convergent power series, then*

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

Taylor Series vs. Power Series

The **Taylor series** of a function $f(x)$ about a point x_0 is defined as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

Thus, a Taylor series is determined by the numbers $\frac{f^{(n)}(x_0)}{n!}$.
OTOH, if $f(x)$ is a function defined by a convergent power series:

$$f(x) \equiv \lim_{N \to \infty} \sum_{n=0}^{N} a_n (x - x_0)^n \tag{9}$$

then $f(x)$ is **identical** to its Taylor series; i.e., if $f(x)$ is of the form (9) then

$$\frac{f^{(n)}(x_0)}{n!} = a_n$$

Taylor Series vs. Power Series, Cont'd

To see this, suppose

$$f(x) = \lim_{N\to\infty} \sum_{n=0}^{N} a_n (x - x_0)^n \qquad (8)$$

Then

$$\begin{aligned}
\frac{1}{0!} f(x_0) &= \lim_{N\to\infty} \sum_{n=0}^{N} a_n (x - x_0)^n \Big|_{x=x_0} \\
&= \left(a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \cdots \right) \Big|_{x=x_0} \\
&= a_0 \\
\frac{1}{1!} \frac{df}{dx}(x_0) &= \frac{1}{1!} \lim_{N\to\infty} \sum_{n=0}^{N} n a_n (x - x_0)^{n-1} \Big|_{x=x_0} \\
&= \Big((0) a_0 (x - x_0)^{-1} + (1) a_1 (x - x_0)^0 + (2) a_2 (x - x_0)^1 + \\
&= a_1
\end{aligned}$$

Taylor Series vs. Power Series, Cont'd

and

$$\begin{aligned}
\frac{1}{2!}\frac{d^2 f}{dx^2}(x_0) &= \frac{1}{2!}\lim_{N\to\infty}\sum_{n=0}^{N} n(n-1)\,a_n\,(x-x_0)^{n-2}\bigg|_{x=x_0} \\
&= \frac{1}{2}\Big((0)(-1)\,a_0\,(x-x_0)^{-2} + (1)(0)\,a_1\,(x-x_0)^{-1} \\
&\quad +(2)(1)a_2\,(x-x_0)^0 + (3)(2)\,a_3\,(x-x_0)^1 + \cdots\Big)\Big|_{x=x_0} \\
&= 0 + 0 + a_2 + 0 + 0 + \cdots
\end{aligned}$$

and, similar computations show that

$$\frac{1}{n!}\frac{d^n f}{dx^n}(x_0) = a_n$$

and so the Taylor expansion of the function defined by power series is exactly the same as the original power series.

Example 2

Find a power series solution to

$$\begin{aligned} y'' - y &= 0 \\ y(0) &= 1 \\ y'(0) &= 1 \end{aligned}$$

We shall assume a trial solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

and we'll try to use the the differential equations and the initial conditions to determine the constants a_n.
Once we find all the coefficients a_n, we can regard the initial value problem as having been solved.

Example 2, Cont'd

Imposing the first initial condition on this $y(x)$ at $x = 0$ yields

$$\begin{aligned} 1 &= y(0) \\ &= a_0 + a_1(0) + a_2(0)^2 + \cdots \\ &= a_0 \end{aligned}$$

so $a_0 = 1$.
Imposing the second initial condition:

$$\begin{aligned} 1 &= y'(0) \\ &= \frac{d}{dx}\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right)\Big|_{x=0} \\ &= \left(0 + a_1 + 2a_2 x + 3a_3 x^2 + \cdots\right)\big|_{x=0} \\ &= 0 + a_1 + 0 + 0 + \cdots \\ &= a_1 \end{aligned}$$

so $a_1 = 1$.

Example 2, Cont'd

Now we still need to determine the coefficients $a_2, a_3, \ldots$. However, this time, rather than differentiating the differential equation to determine $y'''(0)$, $y^{(iv)}(0), \ldots$, we shall determine the coefficients $a_n = \frac{f^{(n)}(0)}{n!}$ by imposing the differential equation directly on the function $y(x) = \sum_{n=0}^{\infty} a_n x^n$.
Substituting $y(x) = \sum_{n=0}^{\infty} a_n x^n$ directly into the differential equation yields

$$\begin{aligned} 0 &= y'' - y \\ &= \frac{d^2}{dx^2}\left(\sum_{n=0}^{\infty} a_n x^n\right) - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} n\,(n-1)\,a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^n \end{aligned}$$

Example 2, Cont'd

Now

$$\begin{aligned}\sum_{n=0}^{\infty} n\,(n-1)\,a_n x^{n-2} &= 0\,(0-1)\,a_0 x^{-2} + (1)\,(0)\,a_1 x^{-1} \\ &\quad + (2)\,(1)\,a_2 x^0 + (3)(2)a_3 x^1 + \cdots \\ &= 0 + 0 + (2)(1)a_2 x^0 + (3)(2)a_3 x^1 \\ &\quad + (4)\,(3)\,a_4 x^2 + \cdots \\ &= \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2} x^n\end{aligned}$$

And so we have

$$\begin{aligned}0 &= \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left[(n+2)\,(n+1)\,a_{n+2} - a_n\right] x^n\end{aligned}$$

Example 2, Cont'd

We have now managed to express the right hand side of the differential equation as a single power series.
From the first basic fact about convergent power series, we know

$$0 = \sum_{n=0}^{\infty} A_n (x - x_0)^n \quad \Rightarrow \quad A_n = 0 \quad \text{for all } n$$

Thus, satisfaction of the differential equation requires

$$(n+2)(n+1)\, a_{n+2} - a_n = 0 \qquad \text{for } n = 0, 1, 2, 3, \ldots$$

or

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

This last set of equations are called the **recursion relations** of the problem. Using them, over and over, we can determine all the coefficients a_n.

Example 2, Cont'd

We have already seen that the initial conditions imply

$$\begin{aligned} y(0) &= 1 \quad \Rightarrow \quad a_0 = 1 \\ y'(0) &= 1 \quad \Rightarrow \quad a_1 = 1 \end{aligned}$$

Example 2, Cont'd

Now we'll begin to apply the recurion relations

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \qquad (RR_n)$$

Now, the recursion relation corresponding to $n = 0$ is

$$RR_{n=0} \quad \Rightarrow \quad a_2 = a_{0+2} = \frac{a_0}{(0+2)(0+1)} = \frac{1}{2 \cdot 1}$$

and similarly,

$$RR_{n=1} \quad \Rightarrow \quad a_3 = a_{1+2} = \frac{a_1}{(1+2)(1+1)} = \frac{1}{3 \cdot 2}$$

$$RR_{n=2} \quad \Rightarrow \quad a_4 = a_{2+2} = \frac{a_2}{(2+2)(2+1)} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1}$$

Example 2, Cont'd

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Now, the recursion relation corresponding to $n = 0$ is

$$RR_{n=0} \quad \Rightarrow \quad a_2 = a_{0+2} = \frac{a_0}{(0+2)(0+1)} = \frac{1}{2 \cdot 1} = \frac{1}{2!}$$

and similarly,

$$RR_{n=1} \quad \Rightarrow \quad a_3 = a_{1+2} = \frac{a_1}{(1+2)(1+1)} = \frac{1}{3 \cdot 2} = \frac{1}{3!}$$

$$RR_{n=2} \quad \Rightarrow \quad a_4 = a_{2+2} = \frac{a_2}{(2+2)(2+1)} = \frac{1}{4 \cdot 3 \cdot 2 \cdot 1} = \frac{1}{4!}$$

Example 2, Cont'd

In fact, this pattern continues and we find

$$a_n = \frac{1}{n!} \qquad \text{for all } n$$

We can thus conclude

$$y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

and so our solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$