Math 2233 - Lecture 16

Agenda

1. 2nd Order Linear ODEs (general case)
2. Taylor Series Method
3. Power Series Method

Review: Solving $y'' + p(x)y' + q(x)y = g(x)$

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The principal goal for the rest of the course will be to develop a means of attacking the general case of Eq. (0) head on.

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(Remark: a function $f(x)$ is **smooth** at a point x if f is continuous at x and all of the derivatives $\dfrac{d^n f}{dx^n}$ are defined at the point x.)

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Theorem (Taylor Expansion Theorem)

Let $f(x)$ be a function that is smooth within a neighborhood of the point x_0. Then

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In Calculus II, Taylor's Theorem is motivated mostly by goal of finding a easy way to approximate an arbitrary function of x by a finite polynomial.
However, here in Math 2233, our goal will be to interpret the infinite series on right hand side as a simply another way of writing down a formula for the function $f(x)$.

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then it is possible to determine all the numbers $\frac{f^{(n)}(x_0)}{n!}$ explicitly. In fact, we will solve differential equations of the form (0) by determining the Taylor expansions of solutions.

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Find the first five terms of the Taylor expansion about $x = 0$ of the solution to

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To make this explicit, we need to figure out numerical values for $y(0), y'(0), y''(0), \ldots$. Now the values of $y(0)$ and $y'(0)$ are determined by the initial conditions

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Example, Cont'd

To determine $y''(0)$, we can evaluate the differential equation itself at $x = 0$:

$$\begin{aligned} y''(0) + 2xy'(0) + y(0) &= 0 \\ \Longrightarrow \quad y''(0) &= (-2xy' - y)\,|_{x=0}) = 0 - y(0) \end{aligned}$$

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$\square$

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- the constants a_n are called the **coefficients** of the power series
- the number x_0 is referred to as the **expansion point**
- the expressions $a_n \left(x - x_0\right)^n$ are the individual **terms** of the power series

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We'll then plug this power series expression for $y(x)$ into the differential equation and try to determine the coefficients a_n by demanding that (8) actually solves the ODE.

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$$f(x) \equiv \lim_{N\to\infty} \sum_{n=0}^{N} a_n (x - x_0)^n$$

Some Basic Facts about Convergent Power Series, Cont'd

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(ii) *$f(x)$ is differentiable and*

$$\frac{df}{dx}(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$

is another convergent power series.

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(iii) *If* $g(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n$ *is another convergent power series, then*

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x - x_0)^n$$

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and so the Taylor expansion of the function defined by power series is exactly the same as the original power series.

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Once we find all the coefficients a_n, we can regard the initial value problem as having been solved.

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Example 2, Cont'd

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$$\begin{aligned}\sum_{n=0}^{\infty} n\,(n-1)\,a_n x^{n-2} &= 0\,(0-1)\,a_0 x^{-2} + (1)\,(0)\,a_1 x^{-1} \\ &\quad + (2)\,(1)\,a_2 x^0 + (3)(2)a_3 x^1 + \cdots \\ &= 0 + 0 + (2)(1)a_2 x^0 + (3)(2)a_3 x^1 \\ &\quad + (4)\,(3)\,a_4 x^2 + \cdots\end{aligned}$$

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$$\begin{aligned}\sum_{n=0}^{\infty} n\,(n-1)\,a_n x^{n-2} &= 0\,(0-1)\,a_0 x^{-2} + (1)\,(0)\,a_1 x^{-1} \\ &\quad + (2)\,(1)\,a_2 x^{0} + (3)(2)a_3 x^{1} + \cdots \\ &= 0 + 0 + (2)(1)a_2 x^{0} + (3)(2)a_3 x^{1} \\ &\quad + (4)\,(3)\,a_4 x^{2} + \cdots \\ &= \sum_{n=0}^{\infty} (n+2)\,(n+1)\,a_{n+2} x^{n}\end{aligned}$$

Example 2, Cont'd

Now

$$\begin{aligned}
\sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2} &= 0(0-1)\, a_0 x^{-2} + (1)(0)\, a_1 x^{-1} \\
&\quad + (2)(1)\, a_2 x^0 + (3)(2) a_3 x^1 + \cdots \\
&= 0 + 0 + (2)(1) a_2 x^0 + (3)(2) a_3 x^1 \\
&\quad + (4)(3)\, a_4 x^2 + \cdots \\
&= \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n
\end{aligned}$$

And so we have

$$0 = \sum_{n=0}^{\infty} (n+2)(n+1)\, a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n$$

Example 2, Cont'd

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$$\begin{aligned}0 &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} \left[(n+2)(n+1)a_{n+2} - a_n\right] x^n\end{aligned}$$

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or

$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \qquad , \qquad n = 0, 1, 2, 3, \ldots$$

This last set of equations are called the **recursion relations** of the problem. Using them, over and over, we can determine all the coefficients a_n.

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We have already seen that the initial conditions imply

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$$\begin{aligned} y(0) &= 1 \quad \Rightarrow \quad a_0 = 1 \\ y'(0) &= 1 \quad \Rightarrow \quad a_1 = 1 \end{aligned}$$

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$$a_{n+2} = \frac{a_n}{(n+2)(n+1)} \qquad (RR_n)$$

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In fact, this pattern continues and we find

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and so our solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$